压电体表面金属电极脱层的屈曲分析 1)

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摘要 基于弹性有限变形理论和电弹性体偏场理论，对半无限压电体及其表面电极层间存在穿透脱层的屈曲问题进行了分析. 采用平面应变模型, 在脱层远处作用有平行于脱层的应变载荷. 使用 Fourier 积分变换, 应用脱层界面的连续条件和电极表面的边界条件将问题归为第 2 类 Cauchy 型奇异积分方程组. 利用 Gauss-Chebyshev 积分公式将奇异积分方程组变为齐次线性代数方程组，以确定临界应变载荷. 通过数值算例，给出了底层为 PZT-4 材料、电极为金属 Pt 在不同的脱层长厚比时的临界应变载荷和屈曲形状，分析了压电体的压电、介电效应对屈曲载荷的影响. 另外给出了脱层屈曲时，脱层尖端奇异性振荡因子随不同脱层长厚比的关系曲线.

关键词 电极脱层，屈曲， Fourier 积分变换，奇异积分方程，临界应变载荷

中图分类号： O343 **文献标识码：** A **文章编号：** 0459-1879(2009)04-0489-14

引 言

随着现代智能材料的快速发展，许多功能器件在现实生活中得到了很好的应用. 红外探测器的微桥像元结构即为一种多层结构：中间层为压电材料或热释电材料，上下表面有很薄的金属电极，上电极表面有热吸收层，同时还有保护层. 在这类多层结构中，电极脱层会造成接触不良，影响结构的热传导效率, 降低结构的响应和探测效果, 使接收到的信号失真. 工程中的层状复合材料结构因制造或使用时会出现层间界面脱层，脱层的局部屈曲会造成结构的承压能力显著降低，影响结构的正常使用. 因此对存在界面脱层的层状结构在受压时的力学行为进行分析对于微智能器件、复合材料结构的设计很有意义.

一些研究者用传统梁和板的结构力学理论分析了脱层的屈曲载荷 [1,2]. 李跃宇 [3] 用基于一阶剪切层板理论的几何非线性有限元分析了压弯载荷作用下含穿透脱层层板的屈曲临界载荷问题. Yeh 等 [4] 考虑了脱层间的接触效应，用试验和有限元方法分析了脱层复合板的弯曲行为. Wee 等 [5] 应用 Euler-Bernoulli 梁和层板理论，用解析和数值方法分析了脱层复合梁的屈曲载荷. Liu 等 [6] 用有限元方法研究了薄层中界面脱层和屈曲问题. Lee 等 [7] 用有限元对有脱层的层状复合结构的屈曲问题进行了分析. Wang 等 [8] 基于弹性理论分析了平行于弹性体表面的裂纹在压缩载荷作用下的失稳问题. Wang 等 [9] 研究了半无限弹性层与表面层间的界面裂纹在载荷作用下的局部屈曲问题. Parlapalli 等 [10] 用 Euler-Bernoulli 梁理论分析了非重叠的脱层复合梁的屈曲问题. Loboda 等 [11] 研究了有限厚度层状体系的双材料界面裂纹脱层的局部失稳问题.

以上研究都是结构材料中的脱层及屈曲问题，而对于智能结构中的脱层和屈曲问题还很少有人研究. 本文对半无限压电体表面金属电极脱层的屈曲问题进行了分析，给出了不同电极脱层长厚比时的临界应变载荷和脱层的屈曲形状，同时考虑了底层压电体的机电耦合效应对临界载荷和脱层尖端奇异振荡因子的影响. 并将分析结果与有限元结果进行了比较.

1 脱层屈曲分析模型

当压电材料层比其表面金属电极层厚得多时，可将压电层看成半无限体. 电极与压电层间的穿透脱层长度为 $2a$, 电极厚度为 h; 按平面应变问题分析 (见图 1). 采用直角坐标系 xoz, 压电体沿 z 轴极化. 在平行于脱层平面的远处作用有均匀压缩应变，当压缩应变达到某一临界值 ε_0 时，电极薄膜脱层将发

2007–12–04 收到第 1 稿， 2008–08–26 收到修改稿.

1) 教育部博士点基金资助项目 (20050247004).

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生屈曲. 电极脱层发生屈曲临界状态的各物理量用上标 “0” 表示. 脱层屈曲时, 电极的位移增量为 u_{1x}, u_{1z}, 应力增量为 σ_{1x}, σ_{1z}, σ_{1zx}. 压电体的位移增量为 u_{2x}, u_{2z}, 应力增量为 σ_{2x}, σ_{2z}, σ_{2zx}, 电势增量为 ϕ, 电位移增量为 D_{2x}, D_{2z}, 各增量均为 x, z 坐标的函数.

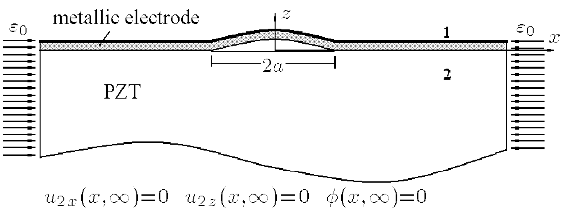


图 1 电极脱层屈曲示意图

Fig.1 Sketch of the electrode delamination and buckling

2 金属电极层的控制方程

根据各向同性弹性体的有限变形理论，电极层的平衡微分方程为

$$\left.\begin{aligned}(1+\varepsilon_{1x}^0)\sigma_{1x,x}+(1+\varepsilon_{1x}^0)\sigma_{1xz,z}+\sigma_{1x}^0u_{1x,xx}=0\\(1+\varepsilon_{1z}^0)\sigma_{1zx,x}+(1+\varepsilon_{1z}^0)\sigma_{1z,z}+\sigma_{1x}^0u_{1z,xx}=0\end{aligned}\right\}\tag{1}$$

式中 $\varepsilon_{1x}^0=\varepsilon_0$, $\varepsilon_{1z}^0=\mu\varepsilon_0/(\mu-1)$, $\sigma_{1x}^0=E\varepsilon_0/(1-\mu^2)$, E 为弹性模量, μ 为泊松比, 剪切模量 $G=E/[2(1+\mu)]$. 记 $E_1=E/[(1+\mu)(1-2\mu)]$, 本构关系为

$$\left.\begin{aligned}\sigma_{1x}&=E_1[(1-\mu)\varepsilon_{1x}+\mu\varepsilon_{1z}]\\\sigma_{1z}&=E_1[(1-\mu)\varepsilon_{1z}+\mu\varepsilon_{1x}]\\\sigma_{1zx}&=G\varepsilon_{1zx}\end{aligned}\right\}\tag{2}$$

应变与位移关系为

$$\left.\begin{aligned}\varepsilon_{1x}&=(1+\varepsilon_{1x}^0)u_{1x,x}\\\varepsilon_{1z}&=(1+\varepsilon_{1z}^0)u_{1z,z}\\\varepsilon_{1zx}&=(1+\varepsilon_{1x}^0)u_{1x,z}+(1+\varepsilon_{1z}^0)u_{1z,x}\end{aligned}\right\}\tag{3}$$

将式 (3) 代入式 (2)，再代入式 (1)，得 u_{1x}, u_{1z} 组成的微分方程组

$$\left.\begin{aligned}g_{11}u_{1x,xx}+g_{12}u_{1x,zz}+g_{13}u_{1z,xz}=0\\g_{21}u_{1x,xz}+g_{22}u_{1z,xx}+g_{23}u_{1z,zz}=0\end{aligned}\right\}\tag{4}$$

式中 g_{ij} 由材料常数和临界应变 ε_0 确定，见附录 A.

应用 Fourier 正弦和余弦的积分变换及逆变换

$$U_{1x}(t,z)=\int_0^\infty u_{1x}(x,z)\sin tx\mathrm{d}x\Rightarrow$$

$$u_{1x}(x,z)=\frac{2}{\pi}\int_0^\infty U_{1x}(t,z)\sin tx\mathrm{d}t$$

$$U_{1z}(t,z)=\int_0^\infty u_{1z}(x,z)\cos tx\mathrm{d}x\Rightarrow$$

$$u_{1z}(x,z)=\frac{2}{\pi}\int_0^\infty U_{1z}(t,z)\cos tx\mathrm{d}t$$

式 (4) 变为

$$\left.\begin{aligned}-g_{11}t^2U_{1x}+g_{12}U_{1x,zz}-g_{13}tU_{1z,z}=0\\g_{21}tU_{1x,z}-g_{22}t^2U_{1z}+g_{23}U_{1z,zz}=0\end{aligned}\right\}\tag{5}$$

设 U_{1x}, U_{1z} 解的形式为

$$\{U_{1x}(t,z)\ \ U_{1z}(t,z)\}^{\mathrm{T}}=\{U_{1x}^0\ \ U_{1z}^0\}^{\mathrm{T}}\cdot\mathrm{e}^{\alpha tz}\tag{6}$$

将式 (6) 代入式 (5)，得

$$\begin{bmatrix}-g_{11}+g_{12}\alpha^2 & -g_{13}\alpha\\ g_{21}\alpha & -g_{22}+g_{23}\alpha^2\end{bmatrix}\begin{Bmatrix}U_{1x}^0\\U_{1z}^0\end{Bmatrix}=0\tag{7}$$

上式要求 U_{1x}^0, U_{1z}^0 有非平凡解，则要求其系数行列式为 0，由此得特征方程

$$R_{14}\alpha^4+R_{12}\alpha^2+R_{10}=0\tag{8}$$

式中， $R_{14}=g_{12}g_{23}$, $R_{12}=g_{13}g_{21}-g_{12}g_{22}-g_{11}g_{23}$, $R_{10}=g_{11}g_{22}$. 解式 (8)，特征根 α 的值由电极的泊松比 μ 和临界应变 ε_0 确定. 对于不同的电极材料常数和不同的临界应变 ε_0, α 值有以下形式:

(1) 4 个特征根均为实数, 表示为 $\alpha_{1i}(i=1\sim4)$, 则位移、应力的表达式可写成

$$\boldsymbol{U}_1=\frac{2}{\pi}\int_0^\infty\Big(\sum_{i=1}^4\boldsymbol{\varGamma}_1A_{1i}\mathrm{e}^{\alpha_{1i}tz}\Big)\boldsymbol{F}_1\mathrm{d}t\tag{9}$$

式中， $\boldsymbol{U}_1=\{u_{1x}\ \ u_{1z}\ \ \sigma_{1x}\ \ \sigma_{1z}\ \ \sigma_{1zx}\}^{\mathrm{T}}$, $\boldsymbol{\varGamma}_1=\{1\ \ \varGamma_{1i}\ \ t\varGamma_{2i}\ \ t\varGamma_{3i}\ \ t\varGamma_{4i}\}^{\mathrm{T}}$, $\boldsymbol{F}_1=\{\sin tx\ \ \cos tx\ \ \cos tx\ \ \cos tx\ \ \sin tx\}^{\mathrm{T}}$, $\varGamma_{ki}$ $(k,\ i=1\sim4)$ 见附录 B, $A_{1i}(i=1\sim4)$ 待定.

(2) 当特征根为 2 对共轭复根，分别表示成: $\alpha_1\pm\mathrm{i}\beta_1$, $\alpha_2\pm\mathrm{i}\beta_2$ $(\alpha_1>0,\beta_1>0,\alpha_2=-\alpha_1,\beta_2=\beta_1)$. 则位移、应力的表达式可以表示成

$$\boldsymbol{U}_1=\frac{2}{\pi}\int_0^\infty\Big\{\sum_{i=1}^2\mathrm{e}^{\alpha_itz}[\boldsymbol{A}_1\cos(\beta_itz)+\boldsymbol{B}_1\sin(\beta_itz)]\Big\}\boldsymbol{F}_1\mathrm{d}t\tag{10}$$

式中 $\boldsymbol{A}_1 = \{A_{1i} \quad \bar{A}_{1i} \quad t\bar{C}_{1i} \quad t\bar{E}_{1i} \quad t\bar{G}_{1i}\}^{\mathrm{T}}$, $\boldsymbol{B}_1 = \{B_{1i} \quad \bar{B}_{1i} \quad t\bar{D}_{1i} \quad t\bar{F}_{1i} \quad t\bar{H}_{1i}\}^{\mathrm{T}}$, $\bar{K}_{1i} = \xi_{1i}^K A_{1i} + \zeta_{1i}^K B_{1i}$, ξ_{1i}^K, ζ_{1i}^K(K 分别取 A, B, C, D, E, F, G, H) 见附录 C, A_{1i}, B_{1i} $(i=1,2)$ 待定.

3 压电层的控制方程

根据电弹性体偏场理论 [12]，压电体的平衡微分方程为

$$\left.\begin{aligned}
&(1+\varepsilon_{2x}^0)\sigma_{2x,x} + (1+\varepsilon_{2x}^0)\sigma_{2xz,z} + \sigma_{2x}^0 u_{2x,xx} = 0\\
&(1+\varepsilon_{2z}^0)\sigma_{2zx,x} + (1+\varepsilon_{2z}^0)\sigma_{2z,z} + \sigma_{2x}^0 u_{2z,xx} = 0\\
&(1+\varepsilon_{2x}^0)D_{2x,x} + (1+\varepsilon_{2z}^0)D_{2z,z} + D_{2z}^0 u_{2x,zx} +\\
&\qquad D_{2z}^0 u_{2z,zz} = 0
\end{aligned}\right\} \tag{11}$$

式中，$\varepsilon_{2x}^0 = \varepsilon_0$, $\varepsilon_{2z}^0 = -c_{13}\varepsilon_0/c_{33}$, $\sigma_{2x}^0 = (c_{11} - c_{13}^2/c_{33})\varepsilon_0$, $D_{2z}^0 = (e_{31} - c_{13}e_{33}/c_{33})\varepsilon_0$.

平面应变问题中，压电体的本构方程为

$$\begin{Bmatrix} \sigma_{2x} \\ \sigma_{2z} \\ \sigma_{2zx} \\ D_{2x} \\ D_{2z} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} \\ 0 & 0 & c_{44} & e_{15} & 0 \\ 0 & 0 & e_{15} & -\lambda_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & -\lambda_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{2x} \\ \varepsilon_{2z} \\ \varepsilon_{2zx} \\ -E_{2x} \\ -E_{2z} \end{Bmatrix} \tag{12}$$

式中 c_{ij} 为压电体的弹性常数，e_{ij} 为压电常数，λ_{ij} 为介电常数. 应变与位移的关系为

$$\left.\begin{aligned}
&\varepsilon_{2x} = (1+\varepsilon_{2x}^0)u_{2x,x}\\
&\varepsilon_{2z} = (1+\varepsilon_{2z}^0)u_{2z,z}\\
&\varepsilon_{2zx} = (1+\varepsilon_{2x}^0)u_{2x,z} + (1+\varepsilon_{2z}^0)u_{2z,x}
\end{aligned}\right\} \tag{13}$$

电场与电势的关系为

$$E_{2x} = -\phi_{,x}, \quad E_{2z} = -\phi_{,z} \tag{14}$$

将式 (13),(14) 代入式 (12)，再代入式 (11)，得 u_{2x}, u_{2z}, ϕ 组成的微分方程组

$$\left.\begin{aligned}
&b_{11}u_{2x,xx} + b_{12}u_{2x,zz} + b_{13}u_{2z,zx} + b_{14}\phi_{,zx} = 0\\
&b_{21}u_{2x,zx} + b_{22}u_{2z,xx} + b_{23}u_{2z,zz} +\\
&\qquad b_{24}\phi_{,xx} + b_{25}\phi_{,zz} = 0\\
&b_{31}u_{2x,zx} + b_{32}u_{2z,xx} + b_{33}u_{2z,zz} +\\
&\qquad b_{34}\phi_{,xx} + b_{35}\phi_{,zz} = 0
\end{aligned}\right\} \tag{15}$$

式中 b_{ij} 见附录 D.

记 u_{2x}, u_{2z}, ϕ 的 Fourier 积分变换及逆变换为

$$\left.\begin{aligned}
&U_{2x}(t,z) = \int_0^\infty u_{2x}(x,z)\sin tx\mathrm{d}x \Rightarrow\\
&\qquad u_{2x}(x,z) = \frac{2}{\pi}\int_0^\infty U_{2x}(t,z)\sin tx\mathrm{d}t\\
&U_{2z}(t,z) = \int_0^\infty u_{2z}(x,z)\cos tx\mathrm{d}x \Rightarrow\\
&\qquad u_{2z}(x,z) = \frac{2}{\pi}\int_0^\infty U_{2z}(t,z)\cos tx\mathrm{d}t\\
&\varPhi(t,z) = \int_0^\infty \phi(x,z)\cos tx\mathrm{d}x \Rightarrow\\
&\qquad \phi(x,z) = \frac{2}{\pi}\int_0^\infty \varPhi(t,z)\cos tx\mathrm{d}t
\end{aligned}\right\} \tag{16}$$

应用式 (15) 和式 (16) 得 U_{2x}, U_{2z}, $\varPhi$ 组成的微分方程组

$$\left.\begin{aligned}
&-b_{11}t^2U_{2x} + b_{12}U_{2x,zz} - b_{13}tU_{2z,z} - b_{14}t\varPhi_{,z} = 0\\
&b_{21}tU_{2x,z} - b_{22}t^2U_{2z} + b_{23}U_{2z,zz} -\\
&\qquad b_{24}t^2\varPhi + b_{25}\varPhi_{,zz} = 0\\
&b_{31}tU_{2x,z} - b_{32}t^2U_{2z} + b_{33}U_{2z,zz} -\\
&\qquad b_{34}t^2\varPhi + b_{35}\varPhi_{,zz} = 0
\end{aligned}\right\} \tag{17}$$

设式 (17) 解的形式为

$$\begin{aligned}
&\{U_{2x}(z,t) \quad U_{2z}(z,t) \quad \varPhi(z,t)\} =\\
&\qquad \{U_{2x}^0 \quad U_{2z}^0 \quad \varPhi^0\}\mathrm{e}^{rtz}
\end{aligned} \tag{18}$$

将式 (18) 代入式 (17), 得

$$\begin{bmatrix} -b_{11} + b_{12}r^2 & -b_{13}r & -b_{14}r \\ b_{21}r & -b_{22} + b_{23}r^2 & -b_{24} + b_{25}r^2 \\ b_{31}r & -b_{32} + b_{33}r^2 & -b_{34} + b_{35}r^2 \end{bmatrix} \cdot \begin{Bmatrix} U_{2x}^0 \\ U_{2z}^0 \\ \varPhi^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

式 (19) 中 $\{U_{2x}^0 \quad U_{2z}^0 \quad \varPhi^0\}$ 存在非平凡解，要求其系数行列式为 0，得特征方程为

$$R_{26}r^6 + R_{24}r^4 + R_{22}r^2 + R_{20} = 0 \tag{20}$$

式中

$$R_{26}=b_{12}(b_{23}b_{35}-b_{25}b_{33})$$

$$R_{24}=b_{11}(b_{25}b_{33}-b_{23}b_{35})+ b_{12}(b_{25}b_{32}+b_{24}b_{33}-b_{23}b_{34}-b_{22}b_{35})+ b_{13}(b_{21}b_{35}-b_{25}b_{31})+b_{14}(b_{23}b_{31}-b_{21}b_{33})$$

$$R_{22}=b_{11}(b_{22}b_{35}+b_{23}b_{34}-b_{24}b_{33}-b_{25}b_{32})+ b_{12}(b_{22}b_{34}-b_{24}b_{32})+b_{13}(b_{24}b_{31}-b_{21}b_{34})+ b_{14}(b_{21}b_{32}-b_{22}b_{31})$$

$$R_{20}=b_{11}(b_{24}b_{32}-b_{22}b_{34})$$

对于特征方程式 (20)，特征根 r 由压电体的材料常数和临界应变 ε_0 确定，解的形式可能有：

(1) 特征根为 6 个实根，分别记为 $r_1=\alpha_{21}$, $r_2=\alpha_{22}$, $r_3=\alpha_{23}$, $r_4=\alpha_{24}=-\alpha_{21}$, $r_5=\alpha_{25}=-\alpha_{22}$, $r_6=\alpha_{26}=-\alpha_{23}$, 其中 α_{21}, α_{22}, $\alpha_{23}>0$, 则位移和电势可表示成

$$\boldsymbol{U}_2=\frac{2}{\pi}\int_0^{\infty}\left(\sum_{i=1}^{6}\boldsymbol{\Gamma}_2A_{2i}\mathrm{e}^{\alpha_{2i}tz}\right)\boldsymbol{F}_2\mathrm{d}t \tag{21}$$

式中 $\boldsymbol{U}_2=\{u_{2x}\ u_{2z}\ \phi\ \sigma_{2x}\ \sigma_{2z}\ \sigma_{2zx}\ D_{2x}\ D_{2z}\}^{\mathrm{T}}$, $\boldsymbol{\Gamma}_2=\{1\ \xi_{2i}^{(2)}\ \zeta_{2i}^{(2)}\ t\Gamma_{1i}^{(2)}\ t\Gamma_{2i}^{(2)}\ t\Gamma_{3i}^{(2)}\ t\Gamma_{4i}^{(2)}\ t\Gamma_{5i}^{(2)}\}^{\mathrm{T}}$, $\boldsymbol{F}_2=\{\sin tx\ \cos tx\ \cos tx\ \cos tx\ \cos tx\ \sin tx\ \sin tx\ \cos tx\}^{\mathrm{T}}$. $\xi_{2i}^{(2)}$, $\zeta_{2i}^{(2)}$, $\Gamma_{Ji}^{(2)}(J=1\sim 5,\ i=1\sim 6)$ 见附录 E, A_{2i} $(i=1\sim 6)$ 待定.

(2) 特征根为 2 个实根和 2 对共轭复根，分别记为： $r_1=\alpha_{21}$, $r_2=\alpha_2+\mathrm{i}\beta_2$, $r_3=\alpha_2-\mathrm{i}\beta_2$, $r_4=-\alpha_2+\mathrm{i}\beta_2$, $r_5=-\alpha_2-\mathrm{i}\beta_2$, $r_6=\alpha_{22}=-\alpha_{21}$, $(\alpha_2>0,\ \beta_2>0)$.

$$u_{2x}=\frac{2}{\pi}\int_0^{\infty}\{A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[A_{22}\cos(\beta_2tz)+A_{23}\sin(\beta_2tz)]+ \mathrm{e}^{-\alpha_2tz}[A_{24}\cos(\beta_2tz)+A_{25}\sin(\beta_2tz)]+A_{26}\mathrm{e}^{\alpha_{22}tz}\}\sin tx\mathrm{d}t \tag{22a}$$

$$u_{2z}=\frac{2}{\pi}\int_0^{\infty}\{\xi_{21}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[\xi_{33}\cos(\beta_2tz)+\xi_{43}\sin(\beta_2tz)]A_{22}+ \mathrm{e}^{\alpha_2tz}[\xi_{34}\cos(\beta_2tz)+\xi_{44}\sin(\beta_2tz)]A_{23}+\mathrm{e}^{-\alpha_2tz}[\xi_{55}\cos(\beta_2tz)+\xi_{65}\sin(\beta_2tz)]A_{24}+ \mathrm{e}^{-\alpha_2tz}[\xi_{56}\cos(\beta_2tz)+\xi_{66}\sin(\beta_2tz)]A_{25}+\xi_{22}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\cos tx\mathrm{d}t \tag{22b}$$

$$\phi=\frac{2}{\pi}\int_0^{\infty}\{\zeta_{21}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[\zeta_{33}\cos(\beta_2tz)+\zeta_{43}\sin(\beta_2tz)]A_{22}+ \mathrm{e}^{\alpha_2tz}[\zeta_{34}\cos(\beta_2tz)+\zeta_{44}\sin(\beta_2tz)]A_{23}+\mathrm{e}^{-\alpha_2tz}[\zeta_{55}\cos(\beta_2tz)+\zeta_{65}\sin(\beta_2tz)]A_{24}+ \mathrm{e}^{-\alpha_2tz}[\zeta_{56}\cos(\beta_2tz)+\zeta_{66}\sin(\beta_2tz)]A_{25}+\zeta_{22}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\cos tx\mathrm{d}t \tag{22c}$$

式中 $\xi_{2i}^{(2)}$, $\zeta_{2i}^{(2)}$, ξ_{KL}, ζ_{KL} 见附录 F.

应力和电位移的表达式为

$$\sigma_{2x}=\frac{2}{\pi}\int_0^{\infty}t\{\Gamma_{11}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[\Gamma_{13}^{(2)}\cos(\beta_2tz)+\Gamma_{15}^{(2)}\sin(\beta_2tz)]A_{22}+ \mathrm{e}^{\alpha_2tz}[\Gamma_{14}^{(2)}\cos(\beta_2tz)+\Gamma_{16}^{(2)}\sin(\beta_2tz)]A_{23}+\mathrm{e}^{-\alpha_2tz}[\Gamma_{17}^{(2)}\cos(\beta_2tz)+\Gamma_{19}^{(2)}\sin(\beta_2tz)]A_{24}+ \mathrm{e}^{-\alpha_2tz}[\Gamma_{18}^{(2)}\cos(\beta_2tz)+\Gamma_{1(10)}^{(2)}\sin(\beta_2tz)]A_{25}+\Gamma_{12}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\cos tx\mathrm{d}t \tag{22d}$$

$$\sigma_{2z}=\frac{2}{\pi}\int_0^{\infty}t\{\Gamma_{21}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[\Gamma_{23}^{(2)}\cos(\beta_2tz)+\Gamma_{25}^{(2)}\sin(\beta_2tz)]A_{22}+ \mathrm{e}^{\alpha_2tz}[\Gamma_{24}^{(2)}\cos(\beta_2tz)+\Gamma_{26}^{(2)}\sin(\beta_2tz)]A_{23}+\mathrm{e}^{-\alpha_2tz}[\Gamma_{27}^{(2)}\cos(\beta_2tz)+\Gamma_{29}^{(2)}\sin(\beta_2tz)]A_{24}+ \mathrm{e}^{-\alpha_2tz}[\Gamma_{28}^{(2)}\cos(\beta_2tz)+\Gamma_{2(10)}^{(2)}\sin(\beta_2tz)]A_{25}+\Gamma_{22}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\cos tx\mathrm{d}t \tag{22e}$$

$$\sigma_{2zx}=\frac{2}{\pi}\int_0^{\infty}t\{\Gamma_{31}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2tz}[\Gamma_{33}^{(2)}\cos(\beta_2tz)+\Gamma_{35}^{(2)}\sin(\beta_2tz)]A_{22}+ \mathrm{e}^{\alpha_2tz}[\Gamma_{34}^{(2)}\cos(\beta_2tz)+\Gamma_{36}^{(2)}\sin(\beta_2tz)]A_{23}+\mathrm{e}^{-\alpha_2tz}[\Gamma_{37}^{(2)}\cos(\beta_2tz)+\Gamma_{39}^{(2)}\sin(\beta_2tz)]A_{24}+ \mathrm{e}^{-\alpha_2tz}[\Gamma_{38}^{(2)}\cos(\beta_2tz)+\Gamma_{3(10)}^{(2)}\sin(\beta_2tz)]A_{25}+\Gamma_{32}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\sin tx\mathrm{d}t \tag{22f}$$

$$D_{2x}=\frac{2}{\pi}\int_0^\infty t\{\varGamma_{41}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2 tz}[\varGamma_{43}^{(2)}\cos(\beta_2 tz)+\varGamma_{45}^{(2)}\sin(\beta_2 tz)]A_{22}+ \mathrm{e}^{\alpha_2 tz}[\varGamma_{44}^{(2)}\cos(\beta_2 tz)+\varGamma_{46}^{(2)}\sin(\beta_2 tz)]A_{23}+\mathrm{e}^{-\alpha_2 tz}[\varGamma_{47}^{(2)}\cos(\beta_2 tz)+\varGamma_{49}^{(2)}\sin(\beta_2 tz)]A_{24}+ \mathrm{e}^{-\alpha_2 tz}[\varGamma_{48}^{(2)}\cos(\beta_2 tz)+\varGamma_{4(10)}^{(2)}\sin(\beta_2 tz)]A_{25}+\varGamma_{42}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\sin tx\mathrm{d}t \tag{22g}$$

$$D_{2z}=\frac{2}{\pi}\int_0^\infty t\{\varGamma_{51}^{(2)}A_{21}\mathrm{e}^{\alpha_{21}tz}+\mathrm{e}^{\alpha_2 tz}[\varGamma_{53}^{(2)}\cos(\beta_2 tz)+\varGamma_{55}^{(2)}\sin(\beta_2 tz)]A_{22}+ \mathrm{e}^{\alpha_2 tz}[\varGamma_{54}^{(2)}\cos(\beta_2 tz)+\varGamma_{56}^{(2)}\sin(\beta_2 tz)]A_{23}+\mathrm{e}^{-\alpha_2 tz}[\varGamma_{57}^{(2)}\cos(\beta_2 tz)+\varGamma_{59}^{(2)}\sin(\beta_2 tz)]A_{24}+ \mathrm{e}^{-\alpha_2 tz}[\varGamma_{58}^{(2)}\cos(\beta_2 tz)+\varGamma_{5(10)}^{(2)}\sin(\beta_2 tz)]A_{25}+\varGamma_{52}^{(2)}A_{26}\mathrm{e}^{\alpha_{22}tz}\}\cos tx\mathrm{d}t \tag{22h}$$

式中 $\varGamma_{NM}^{(2)}$ $(N=1\sim 5,\ M=1\sim 10)$ 由材料常数和临界应变 ε_0 确定，见附录 G. 对半无限压电体而言，当 $z\to-\infty$ 时，要求其位移、电势、应力、电位移均应趋于 0，可知 $A_{24}=A_{25}=A_{26}=0$. 式 (9),(10) 中的 $A_{1i}(i=1\sim 4)$，式 (21)，(22a)~(22h) 中的 A_{2j} $(j=1\sim 3)$ 均为积分变量 t 的函数.

4 奇异积分方程

在金属电极脱层屈曲时的临界状态，边界条件及界面连续条件为：

电极表面： $z=h$

$$\sigma_{1z}(x,h)=0,\ \ \sigma_{1zx}(x,h)=0,\ \ |x|<\infty \tag{23a}$$

脱层界面： $z=0$

$$\left.\begin{aligned}&\sigma_{1z}(x,0)=\sigma_{2z}(x,0),\ \ \sigma_{1zx}(x,0)=\sigma_{2zx}(x,0),\\&|x|<\infty\end{aligned}\right\} \tag{23b}$$

$$\sigma_{1z}(x,0)=0,\ \ \sigma_{1zx}(x,0)=0,\ \ |x|<a \tag{23c}$$

$$\left.\begin{aligned}&u_{1x}(x,0)=u_{2x}(x,0),\ \ u_{1z}(x,0)=u_{2z}(x,0),\\&|x|>a\end{aligned}\right\} \tag{23d}$$

$$D_{2z}(x,0)=0,\ \ |x|<\infty \tag{23e}$$

$A_{1i}(i=1\sim 4)$，$A_{2j}(j=1\sim 3)$ 由上述边界和界面条件确定.

对于金属电极和压电体所得到的特征方程，其特征根取决于相应的材料常数和临界应变 ε_0，因此由上述边界和界面条件确定的方程组有以下 4 种可能组合：

(1) 电极： 4 个实根；压电体： 6 个实根.

(2) 电极： 4 个实根；压电体： 2 个实根和 2 对共轭复根.

(3) 电极： 2 对共轭复根；压电体： 6 个实根.

(4) 电极： 2 对共轭复根；压电体： 2 个实根和 2 对共轭复根.

现考虑情形 (1)，其它的 3 种情形同理.

由式 (23a) 得

$$\varGamma_{31}A_{11}\mathrm{e}^{\alpha_{11}th}+\varGamma_{32}A_{12}\mathrm{e}^{\alpha_{12}th}+\varGamma_{33}A_{13}\mathrm{e}^{\alpha_{13}th}+ \varGamma_{34}A_{14}\mathrm{e}^{\alpha_{14}th}=0 \tag{24a}$$

$$\varGamma_{41}A_{11}\mathrm{e}^{\alpha_{11}th}+\varGamma_{42}A_{12}\mathrm{e}^{\alpha_{12}th}+\varGamma_{43}A_{13}\mathrm{e}^{\alpha_{13}th}+ \varGamma_{44}A_{14}\mathrm{e}^{\alpha_{14}th}=0 \tag{24b}$$

由式 (23b) 得

$$\varGamma_{31}A_{11}+\varGamma_{32}A_{12}+\varGamma_{33}A_{13}+\varGamma_{34}A_{14}-\varGamma_{21}^{(2)}A_{21}- \varGamma_{22}^{(2)}A_{22}-\varGamma_{23}^{(2)}A_{23}=0 \tag{24c}$$

$$\varGamma_{41}A_{11}+\varGamma_{42}A_{12}+\varGamma_{43}A_{13}+\varGamma_{44}A_{14}-\varGamma_{31}^{(2)}A_{21}- \varGamma_{32}^{(2)}A_{22}-\varGamma_{33}^{(2)}A_{23}=0 \tag{24d}$$

由式 (23e) 得

$$\varGamma_{51}^{(2)}A_{21}+\varGamma_{52}^{(2)}A_{22}+\varGamma_{53}^{(2)}A_{23}=0 \tag{24e}$$

对式 (23d)，引入位错密度函数

$$\left.\begin{aligned}f_u(x)&=\frac{\mathrm{d}}{\mathrm{d}x}[u_{1x}(x,0)-u_{2x}(x,0)]\\f_w(x)&=\frac{\mathrm{d}}{\mathrm{d}x}[u_{1z}(x,0)-u_{2z}(x,0)]\end{aligned}\right\} \tag{24f}$$

可得

$$A_{11}+A_{12}+A_{13}+A_{14}-A_{21}-A_{22}-A_{23}=F_u(t) \tag{24g}$$

$$\varGamma_{11}A_{11}+\varGamma_{12}A_{12}+\varGamma_{13}A_{13}+\varGamma_{14}A_{14}-\xi_{21}^{(2)}A_{21}- \xi_{22}^{(2)}A_{22}-\xi_{23}^{(2)}A_{23}=F_w(t) \tag{24h}$$

式中

$$F_u(t)=\frac{1}{t}\int_0^a f_u(\xi)\cos t\xi\mathrm{d}\xi$$

$$F_w(t)=-\frac{1}{t}\int_0^a f_w(\xi)\sin t\xi\mathrm{d}\xi$$

将式 (24a)~(24e)，(24g)，(24h) 等 7 个方程写成矩阵形式

$$\begin{bmatrix} \Gamma_{31}e^{\alpha_{11}th} & \Gamma_{32}e^{\alpha_{12}th} & \Gamma_{33}e^{\alpha_{13}th} & \Gamma_{34}e^{\alpha_{14}th} & 0 & 0 & 0 \\ \Gamma_{41}e^{\alpha_{11}th} & \Gamma_{42}e^{\alpha_{12}th} & \Gamma_{43}e^{\alpha_{13}th} & \Gamma_{44}e^{\alpha_{14}th} & 0 & 0 & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & -\Gamma_{21}^{(2)} & -\Gamma_{22}^{(2)} & -\Gamma_{23}^{(2)} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & -\Gamma_{31}^{(2)} & -\Gamma_{32}^{(2)} & -\Gamma_{33}^{(2)} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & -\xi_{21}^{(2)} & -\xi_{22}^{(2)} & -\xi_{23}^{(2)} \\ 0 & 0 & 0 & 0 & \Gamma_{51}^{(2)} & \Gamma_{52}^{(2)} & \Gamma_{53}^{(2)} \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \\ A_{21} \\ A_{22} \\ A_{23} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F_u(t) \\ F_w(t) \\ 0 \end{bmatrix} \tag{25}$$

式 (25) 可简记为矩阵形式

$$\boldsymbol{\Delta A} = \boldsymbol{F} \tag{26}$$

$\boldsymbol{\Delta}$ 为式 (25) 的系数矩阵. 记式 (26) 系数矩阵的第 i 行、第 j 列的代数余子式为 Δ_{ij}. 求解上式可得 A_{1k} $(k=1\sim4)$ 关于 $F_u(t)$, $F_w(t)$ 的表达式.

$$A_{1k}(t) = [|\Delta_{5k}|F_u(t) + |\Delta_{6k}|F_w(t)]/|\boldsymbol{\Delta}|\,,\ (k=1\sim4) \tag{27}$$

由式 (23c) 得

$$\left.\begin{aligned} &\frac{2}{\pi}\int_0^\infty t[\Gamma_{31}A_{11}(t)+\Gamma_{32}A_{12}(t)+\Gamma_{33}A_{13}(t)+\Gamma_{34}A_{14}(t)]\cos tx\mathrm{d}t = 0\,, \\ &\frac{2}{\pi}\int_0^\infty t[\Gamma_{41}A_{11}(t)+\Gamma_{42}A_{12}(t)+\Gamma_{43}A_{13}(t)+\Gamma_{44}A_{14}(t)]\sin tx\mathrm{d}t = 0\,, \end{aligned} \quad |x|<a \right\} \tag{28}$$

将式 (27) 代入式 (28)，整理得

$$\frac{2}{\pi}\int_0^\infty \left[Q_{11}(t)\int_0^a f_u(\xi)\cos t\xi\mathrm{d}\xi + Q_{12}(t)\int_0^a f_w(\xi)\sin t\xi\mathrm{d}\xi\right]\cos tx\mathrm{d}t = 0 \tag{29a}$$

$$\frac{2}{\pi}\int_0^\infty \left[Q_{21}(t)\int_0^a f_u(\xi)\cos t\xi\mathrm{d}\xi + Q_{22}(t)\int_0^a f_w(\xi)\sin t\xi\mathrm{d}\xi\right]\sin tx\mathrm{d}t = 0 \tag{29b}$$

式中

$$\begin{aligned} &Q_{11}(t) = \Big(\sum_{i=1}^4 \Gamma_{3i}|\Delta_{5i}|\Big)\Big/|\boldsymbol{\Delta}|, \quad Q_{12}(t) = -\Big(\sum_{i=1}^4 \Gamma_{3i}|\Delta_{6i}|\Big)\Big/|\boldsymbol{\Delta}|, \\ &Q_{21}(t) = \Big(\sum_{i=1}^4 \Gamma_{4i}|\Delta_{5i}|\Big)\Big/|\boldsymbol{\Delta}|, \quad Q_{22}(t) = -\Big(\sum_{i=1}^4 \Gamma_{4i}|\Delta_{6i}|\Big)\Big/|\boldsymbol{\Delta}|, \end{aligned} \quad (i=1\sim4)$$

整理式 (29a), (29b) 得奇异积分方程组

$$\beta_1 f_u(x) + \frac{1}{\pi}\int_{-a}^a \frac{f_w(\xi)}{\xi-x}\mathrm{d}\xi + \frac{1}{\pi}\int_{-a}^a \big[f_u(\xi)K_{11}(\xi,x) + f_w(\xi)K_{12}(\xi,x)\big]\mathrm{d}\xi = 0 \tag{30a}$$

$$-\beta_2 f_w(x) + \frac{1}{\pi}\int_{-a}^a \frac{f_u(\xi)}{\xi-x}\mathrm{d}\xi + \frac{1}{\pi}\int_{-a}^a \big[f_u(\xi)K_{21}(\xi,x) + f_w(\xi)K_{22}(\xi,x)\big]\mathrm{d}\xi = 0 \tag{30b}$$

$f_u(\xi)$, $f_w(\xi)$ 除了应满足式 (30a),(30b) 之外，还应满足位移单值条件

$$\frac{1}{\pi}\int_{-a}^a f_u(\xi)\mathrm{d}\xi = 0\,, \quad \frac{1}{\pi}\int_{-a}^a f_w(\xi)\mathrm{d}\xi = 0\,, \quad |x|<a \tag{30c}$$

式中

$$K_{11}(\xi,x) = \frac{1}{Q_{12}^\infty}\int_0^\infty [Q_{11}(t)-Q_{11}^\infty]\cos t\xi\cos tx\mathrm{d}t\,, \quad K_{12}(\xi,x) = \frac{1}{Q_{12}^\infty}\int_0^\infty [Q_{12}(t)-Q_{12}^\infty]\sin t\xi\cos tx\mathrm{d}t$$

$$K_{21}(\xi,x) = -\frac{1}{Q_{21}^\infty}\int_0^\infty [Q_{21}(t)-Q_{21}^\infty]\cos t\xi\sin tx\mathrm{d}t\,, \quad K_{22}(\xi,x) = -\frac{1}{Q_{21}^\infty}\int_0^\infty [Q_{22}(t)-Q_{22}^\infty]\sin t\xi\sin tx\mathrm{d}t$$

$$Q_{ij}^\infty = \lim_{t\to\infty} Q_{ij}(t)\ \ (i,j=1,2)\,, \quad \beta_1 = Q_{11}^\infty/Q_{12}^\infty\,, \quad \beta_2 = Q_{22}^\infty/Q_{21}^\infty$$

5 奇异积分方程的数值求解

对于第 2 类的 Cauchy 型奇异积分方程组 (30a)~(30c), 可利用 Gauss-Chebyshev 积分公式, 离散成一齐次线性代数方程组, 引进变量代换 $\xi = as$, $x = a\zeta$, 将式 (30) 变换为

$$\beta_1 f_u^*(\zeta) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_w^*(s)}{s-\zeta}\mathrm{d}s + \frac{1}{\pi}\int_{-1}^{1}[f_u^*(s)K_{11}^*(s,\zeta) + f_w^*(s)K_{12}^*(s,\zeta)]\mathrm{d}s = 0 \tag{31a}$$

$$-\beta_2 f_w^*(\zeta) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_u^*(s)}{s-\zeta}\mathrm{d}s + \frac{1}{\pi}\int_{-1}^{1}[f_u^*(s)K_{21}^*(s,\zeta) + f_w^*(s)K_{22}^*(s,\zeta)]\mathrm{d}s = 0 \tag{31b}$$

$$\frac{1}{\pi}\int_{-1}^{1} f_u^*(s)\mathrm{d}s = 0, \quad \frac{1}{\pi}\int_{-1}^{1} f_w^*(s)\mathrm{d}s = 0 \qquad (|\zeta| < 1) \tag{31c}$$

式中

$$f_u^*(\zeta) = f_u(a\zeta) = f_u(x), \quad f_w^*(\zeta) = f_w(a\zeta) = f_w(x)$$

$$f_u^*(s) = f_u(as) = f_u(\xi), \quad f_w^*(s) = f_w(as) = f_w(\xi)$$

$$K_{ij}^*(s,\zeta) = aK_{ij}(as, a\zeta) = aK_{ij}(\xi, x)$$

基于 Muskhelishvili[13] 奇异积分方程理论, 式 (31) 中 $f_u^*(s)$, $f_w^*(s)$ 的一般解可取复数形式

$$\left.\begin{aligned} f_u^*(s) + \mathrm{i}f_w^*(s) &= \frac{g_1^*(s) + \mathrm{i}g_2^*(s)}{(1-s^2)^{1/2+\mathrm{i}\gamma}} \\ \gamma &= \frac{1}{2\pi}\ln\left(\frac{1+\sqrt{\beta_1\beta_2}}{1-\sqrt{\beta_1\beta_2}}\right) \end{aligned}\right\} \tag{32}$$

展开式 (32) 右端, 实部和虚部分别为

$$\left.\begin{aligned} f_u^*(s) &= \frac{1}{\sqrt{1-s^2}}[g_1^*(s)\cos\theta - g_2^*(s)\sin\theta] \\ f_w^*(s) &= \frac{1}{\sqrt{1-s^2}}[g_1^*(s)\sin\theta + g_2^*(s)\cos\theta] \end{aligned}\right\} \tag{33}$$

式中 $\theta = -\gamma\ln(1-s^2)$.

将式 (33) 代入式 (31a)~(31c), 同时采用 Gauss-Chebyshev 积分公式

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-s^2}}\frac{\psi(s)}{s-\zeta}\mathrm{d}s \simeq \sum_{i=1}^{n}\frac{1}{n}\frac{\psi(s_i)}{s_i-\zeta_k} \tag{34}$$

$s_i = \cos\left(\frac{2i-1}{2n}\pi\right)$, $\zeta_k = \cos\left(\frac{k}{n}\pi\right)$ $(i = 1, 2, \cdots, n)$, $(k = 1, 2, \cdots, n-1)$.

同时使用下列近似计算

$$\left.\begin{aligned} f_u^*(\zeta_k) &\simeq \frac{1}{2}[f_u^*(s_k) + f_u^*(s_{k+1})] \\ f_w^*(\zeta_k) &\simeq \frac{1}{2}[f_w^*(s_k) + f_w^*(s_{k+1})] \end{aligned}\right\} \tag{35}$$

由式 (31a)~(31c) 可得关于 g_1^* (s_i), g_2^* (s_i) 的 $2n$ 个方程组成的齐次代数方程组

$$\begin{aligned} &\frac{\beta_1}{2}\Big[\frac{\cos\theta_k}{\sqrt{1-s_k^2}}g_1^*(s_k) + \frac{\cos\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}g_1^*(s_{k+1}) - \frac{\sin\theta_k}{\sqrt{1-s_k^2}}g_2^*(s_k) - \frac{\sin\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}g_2^*(s_{k+1})\Big] + \\ &\quad\frac{1}{n}\sum_{i=1}^{n}\Big\{\Big[\frac{\sin\theta_i}{s_i-\zeta_k} + K_{11}^*(s_i,\zeta_k)\cos\theta_i + K_{12}^*(s_i,\zeta_k)\sin\theta_i\Big]g_1^*(s_i) + \\ &\quad\Big[\frac{\cos\theta_i}{s_i-\zeta_k} - K_{11}^*(s_i,\zeta_k)\sin\theta_i + K_{12}^*(s_i,\zeta_k)\cos\theta_i\Big]g_2^*(s_i)\Big\} = 0 \end{aligned} \tag{36a}$$

$$\begin{aligned} &-\frac{\beta_2}{2}\Big[\frac{\sin\theta_k}{\sqrt{1-s_k^2}}g_1^*(s_k) + \frac{\sin\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}g_1^*(s_{k+1}) + \frac{\cos\theta_k}{\sqrt{1-s_k^2}}g_2^*(s_k) + \frac{\cos\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}g_2^*(s_{k+1})\Big] + \\ &\quad\frac{1}{n}\sum_{i=1}^{n}\Big\{\Big[\frac{\cos\theta_i}{s_i-\zeta_k} + K_{21}^*(s_i,\zeta_k)\cos\theta_i + K_{22}^*(s_i,\zeta_k)\sin\theta_i\Big]g_1^*(s_i) + \\ &\quad\Big[-\frac{\sin\theta_i}{s_i-\zeta_k} - K_{21}^*(s_i,\zeta_k)\sin\theta_i + K_{22}^*(s_i,\zeta_k)\cos\theta_i\Big]g_2^*(s_i)\Big\} = 0 \end{aligned} \tag{36b}$$

$$\left.\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}[g_1^*(s_i)\cos\theta_i-g_2^*(s_i)\sin\theta_i]=0\\ \frac{1}{n}\sum_{i=1}^{n}[g_1^*(s_i)\sin\theta_i+g_2^*(s_i)\cos\theta_i]=0\end{aligned}\right\}\quad(36c)$$

式中 $\theta_i=-\gamma\ln(1-s_i^2)$. 将式 (36) 整理成下列齐次线性代数方程组

$$\boldsymbol{MG}=\boldsymbol{0}\quad(37)$$

式中 $\boldsymbol{G}=\{g_1^*(s_1)\quad g_2^*(s_1)\quad g_1^*(s_2)\quad g_2^*(s_2)\quad\cdots\quad g_1^*(s_n)\quad g_2^*(s_n)\}^{\mathrm{T}}$.

屈曲载荷 ε_0 可由式 (37) 的系数行列式为 0 的条件确定，即

$$|\boldsymbol{M}|_{2n\times 2n}=0\quad(38)$$

同时可求特征向量 $\boldsymbol{G}$. 电极脱层的屈曲形状可由下式确定

$$\delta w(x,0)=[u_{1z}(x,0)-u_{2z}(x,0)]=\\ -a\int_x^1 f_w^*(\xi)\mathrm{d}\xi\quad(39)$$

脱层屈曲时 $u_{1z}(x,0)$ 的位移远比 $u_{2z}(x,0)$ 大得多，因此计算电极屈曲形状时 $u_{2z}(x,0)$ 可忽略不计.

6 算例分析

算例中电极为金属 Pt, 厚 $h=0.1\mu$m, 弹性模量 $E=159$ GPa, 泊松比 $\mu=0.28$, 电极脱层长度为 $2a$. 半无限压电体为 PZT-4 材料，且沿 z 轴极化，材料常数见表 1. 用 Mathematica 编程分析. 对于式 (8),(20), 对不同 ε_0 有不同的特征根. 经计算得知此算例中 ε_0 在 0~0.2 内取值时，式 (8) 的特征根为 2 对共轭复根，式 (20) 的特征根为 2 个实根和 2 对共轭复根.

表 1 PZT-4 的材料常数

Table 1 Material properties of PZT-4

Elastic constants/ GPa	Piezoelectric constants/ ($\mathrm{C\cdot m^{-2}}$)	Dielectric constants/ ($\mathrm{F\cdot m^{-1}}$)
$c_{11}=139$	$e_{15}=13.44$	$\lambda_{11}=6.0\times10^{-9}$
$c_{13}=74.3$	$e_{31}=-6.98$	$\lambda_{33}=5.47\times10^{-9}$
$c_{33}=113$	$e_{33}=13.84$	
$c_{44}=25.6$		

采用 Gauss-Chebyshev 积分公式求解奇异积分方程式 (31a)~(31c), 在 [−1,1] 区间离散成 $2n$ 个齐次线性代数方程组式 (36). 当取不同的离散数 n 分别计算在不同脱层长厚比时的临界应变值 ε_0，结果曲线见图 2. 从图中可见，临界应变 ε_0 随离散数 n 的收敛性很好. 因此，文后结果均为 $n=60$ 时的计算结果.

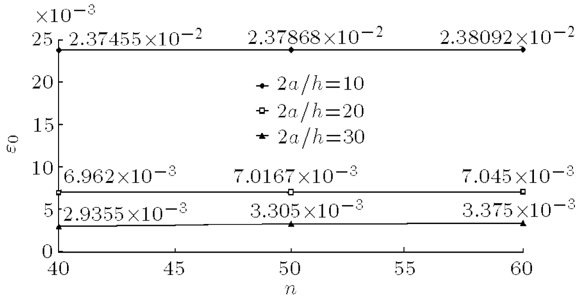


图 2 临界应变随离散数 n 的变化曲线

Fig.2 Curves of the critical strains with respect to the discrete numbers n

如果将脱层 $2a$ 范围内的电极看作是一厚 h，单位宽度 $B=1$ 的梁，分别考虑此梁两端铰支和两端固支的边界条件. 压杆失稳的 Euler 公式为

$$P_{\mathrm{cr}}=\frac{\pi^2EI}{(\varphi l)^2}\quad(40)$$

式中 φ 为长度系数, 当 $\varphi=1$ 对应铰支边界，$\varphi=0.5$ 对应固支边界. E 为压杆的弹性模量，I 为截面惯性矩，l 为压杆长度. P_{cr} 为临界压力. 取 $l=2a$, $I=Bh^3/12$, $P_{\mathrm{cr}}=EBh\varepsilon_{\mathrm{cr}}$, 则可得临界应变为

$$\varepsilon_{\mathrm{cr}}=\frac{\pi^2}{12\varphi^2}\left(\frac{h}{2a}\right)^2\quad(41)$$

如果将 ε_0 看作式 (41) 的 $\varepsilon_{\mathrm{cr}}$, 则电极脱层的等效长度系数 φ_E 为

$$\varphi_E=\frac{\pi}{\sqrt{12\varepsilon_0}}\left(\frac{h}{2a}\right)\quad(42)$$

不同电极脱层长厚比时临界应变 ε_0 的计算结果见图 3, 其中考虑了机电耦合 (electro-mechanical coupling, EMC) 和机电不耦合 (electro-mechanical

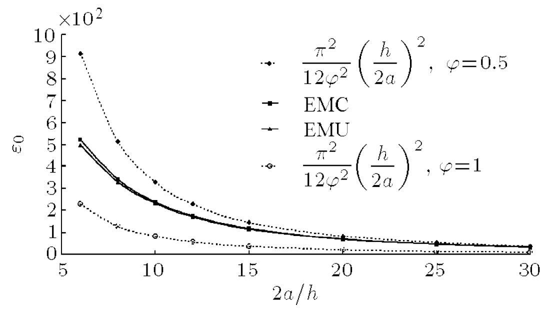


图 3 临界应变 ε_0 随脱层长厚比的关系曲线图

Fig.3 Curves of the critical strains ε_0 with the delamination length to thickness ratios uncoupling, EMU)，即不考虑 PZT-4 的压电和介电效应两种情况．同时，也和根据式 (41) 计算的临界应变曲线进行了比较．

从图 3 的曲线可以看出，电极脱层屈曲时的临界应变在两端铰支与两端固支压杆的临界应变之间，说明计算结果是合理的．实际上，脱层电极两端相当于弹性支承边界．随着脱层长厚比 $2a/h$ 增大，临界应变的结果则逐渐趋于两端固支的情况，这一点还可以从等效长度系数 φ_E 与电极脱层长厚比的关系曲线图 4 中看出．脱层长厚比越大，φ_E 越趋于固支时压杆 Euler 公式的长度系数 0.5.

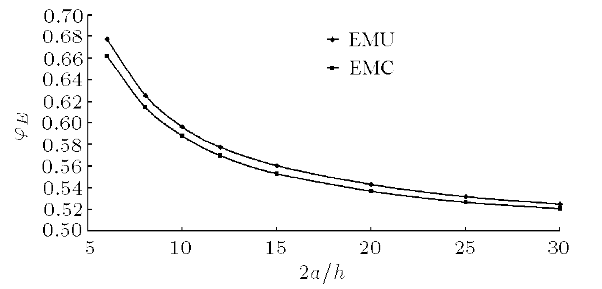


图 4　等效长度系数 φ_E 与电极脱层长厚比的关系曲线图

Fig.4　Relations between the equivalent length coefficient φ_E and the delamination length to thickness ratios

临界应变 ε_0 的结果在考虑半无限压电体的机电耦合和不耦合的两种情况下差别不大，即说明压电体的压电效应对脱层屈曲的临界压缩载荷影响小，特别是当脱层的长厚比 $2a/h$ 较大时，这种影响更小，例如当 $2a/h = 30$ 时，两者的差别为 2.7%．且脱层屈曲时的临界应变 ε_0 随着脱层长厚比 $2a/h$ 的增大而逐渐降低．在脱层长厚比较小时，耦合情况下的临界应变比不耦合情况下的略大，而脱层长厚比较大时两种情况下的结果逐渐趋于一致．

数值结果还给出了电极局部脱层在脱层长厚比为 $2a/h = 10, 20, 30$ 的屈曲构型，见图 5．从脱层的屈曲形状来看，结果是合理的．

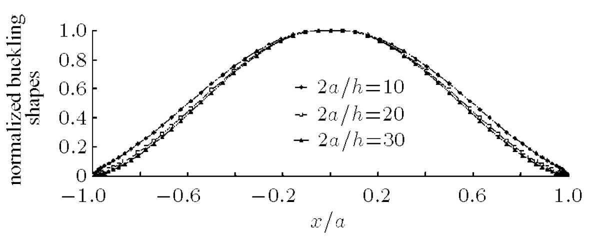
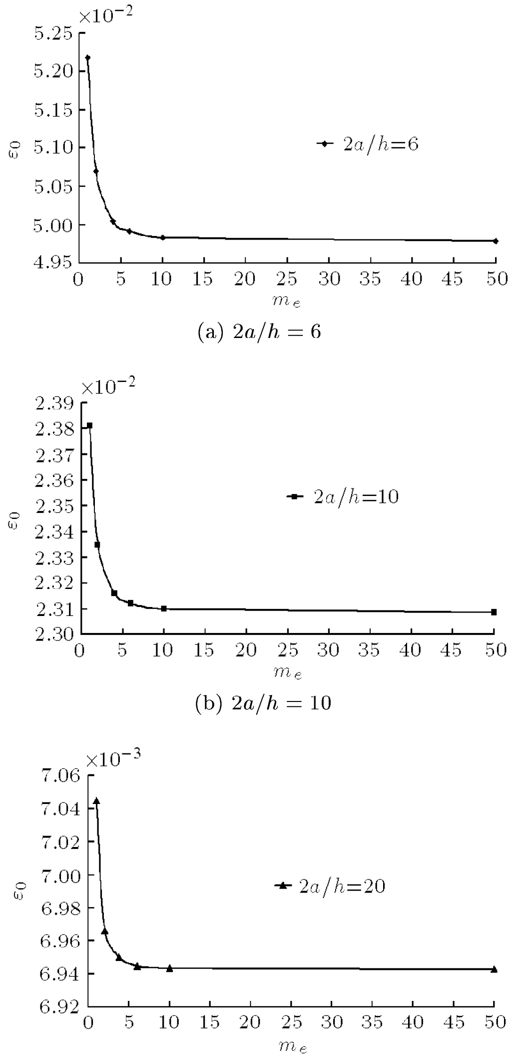


图 5　不同电极脱层长厚比时的屈曲形状

Fig.5　Buckling shapes with respect to different delamination length to thickness ratios

如果将压电系数 e_{ij} 取为原来的 $1/m_e$，分析临界应变 ε_0 与 m_e 的关系．计算了脱层长厚比 $2a/h = 6$, 10, 20 时的 ε_0 与 m_e 的关系曲线，见图 6．从图中曲线可以看出，随着压电系数降低倍数 m_e 的增大，临界应变 ε_0 则迅速降低，随后约在 $m_e > 10$ 之后压电系数的影响基本上就很小了，这说明压电体的压电效应对 ε_0 的影响迅速减小．同时说明了压电体的压电效应对 ε_0 是有影响的．



(a) $2a/h = 6$

(b) $2a/h = 10$

(c) $2a/h = 20$

图 6　临界应变 ε_0 与 m_e 的关系曲线图

Fig.6　Curves of critical strains ε_0 with reduction multiples m_e of the piezoelectric constants

对双材料界面脱层，电极局部屈曲时，脱层尖端的应力强度因子具有振荡性．不同的电极脱层长厚比，脱层屈曲时的临界载荷是不一样的．脱层长厚比小时，屈曲的临界应变 ε_0 大；脱层长厚比大时，临界应变小，因此裂纹尖端奇异振荡因子 γ 在不同

的脱层长厚比是不同的，见图 5 所示.

从图 7 可知，底层 PZT 机电耦合效应对振荡因子 γ 的影响是很大的．而且 γ 随着长厚比 $2a/h$ 增大的趋势则相反，考虑 PZT 的机电耦合效应时振荡因子 γ 随脱层长厚比的增大略有增大，然而不考虑 PZT 的机电耦合效应时则是降低的.

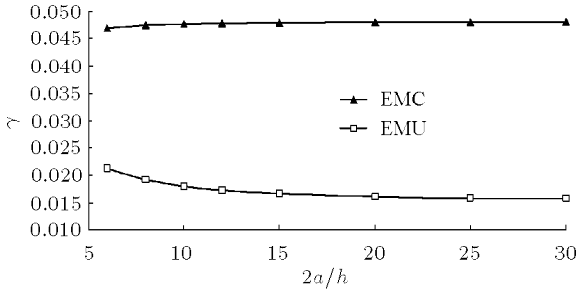


图 7 振荡因子 γ 与电极脱层的长厚比的关系曲线图

Fig.7 Curves of singular oscillating factors γ with respect to delamination length to thickness ratios

为验证前面分析及计算结果的正确性，对该问题用有限元建模进行屈曲分析，因电极脱层的屈曲是发生在局部，当压电层的厚度比电极层厚度大得多时，屈曲对厚度的影响就很小了. 有限元建模时取总厚度比电极厚度为 10/1 进行建模．当 $2a/h = 20$, 30, 40 时，分析结果见图 8. 换算得到的临界应变 ε' 与 ε_0 的结果对比见表 2.

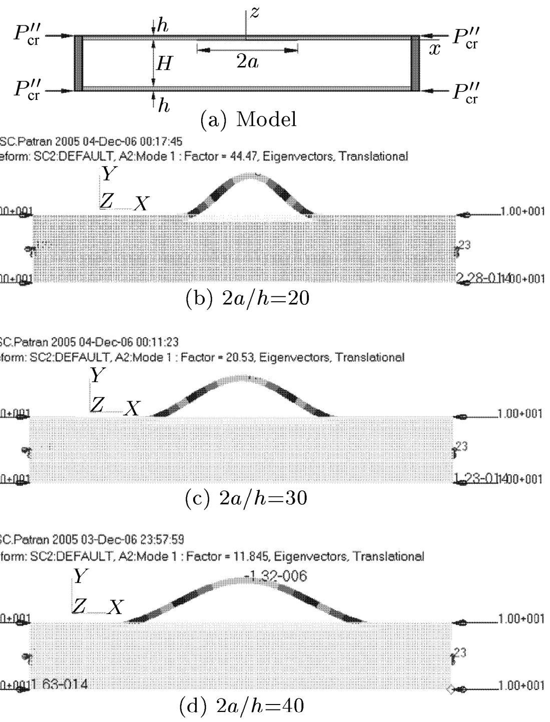


图 8 有限元模型及屈曲分析结果

Fig.8 Finite element model and results of the buckling analysis

表 2 有限元分析的临界应变 ε' 与 ε_0 对比表

Table 2 Comparison between the FEM critical strains ε' and ε_0

$2a/h$	20	30	40
ε'	7.7105×10^{-3}	3.5596×10^{-3}	2.054×10^{-3}
ε_0	7.0449×10^{-3}	3.3749×10^{-3}	1.974×10^{-3}
error	8.6%	5.2%	3.9%

从表 2 的对比结果来看，两者间的误差是合理的．也说明了本文方法是可行的，推导过程和计算结果是正确的．另外还可以从位错密度函数 $f_u^*(\zeta)$, $f_w^*(\zeta)$ 曲线 (图 9) 看出计算结果的正确性． $f_u^*(\zeta)$ 是对称的，而 $f_w^*(\zeta)$ 是反对称的.

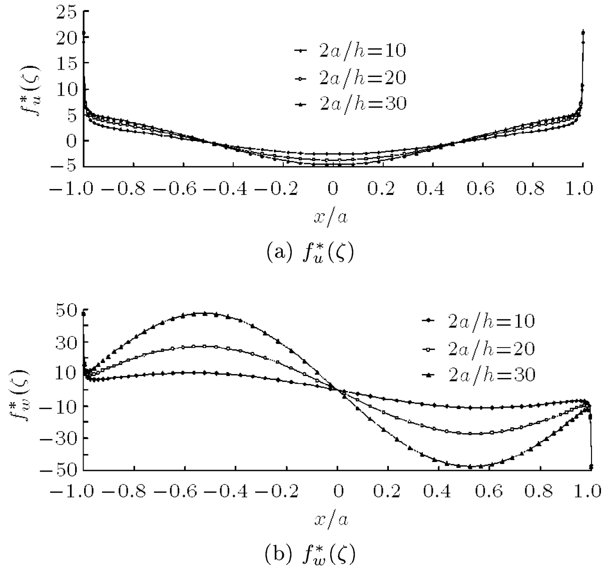


图 9 位错密度函数 $f_u^*(\zeta)$, $f_w^*(\zeta)$ 曲线

Fig.9 Dislocation density function $f_u^*(\zeta)$ and $f_w^*(\zeta)$

7 结 论

本文分析了半无限压电体表面电极脱层的局部屈曲问题．采用平面应变模型，应用有限变形理论和电弹性体的偏场理论，以及相应的界面和边界条件，将问题归为求解奇异积分方程组．基于 Muskhelishvili[13] 的奇异积分方程理论，采用 Gauss-Chebyshev 积分公式将奇异积分方程组化为求解齐次线性代数方程组．通过实例计算压电体为 PZT-4, 电极脱层为 Pt 的体系在均匀应变载荷下不同脱层长厚比时局部脱层的屈曲应变载荷和相应的屈曲形状. 分析了压电体机电耦合效应对屈曲载荷的影响. 计算了双层界面脱层屈曲时，其裂尖奇异性振荡因

子随脱层长厚比的关系．另外给出了位错密度函数 $f_u^*(\zeta)$ 的对称曲线，及 $f_w^*(\zeta)$ 的反对称曲线．所分析的结果是合理的．同时与 FEM 结果进行了对比，两者相差在合理范围内．分析结果对于层状体系智能材料结构的设计有重要的参考意义．

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附录 A

$$g_{11}=\frac{(1+\varepsilon_{1x}^0)^2(1-\mu)}{(1+\mu)(1-2\mu)}+\frac{\varepsilon_{1x}^0}{1-\mu^2}\,,\quad g_{12}=\frac{(1+\varepsilon_{1x}^0)^2}{2(1+\mu)}\,,\quad g_{13}=\frac{(1+\varepsilon_{1x}^0)(1+\varepsilon_{1z}^0)}{2(1+\mu)(1-2\mu)}$$

$$g_{21}=\frac{(1+\varepsilon_{1x}^0)(1+\varepsilon_{1z}^0)}{2(1+\mu)(1-2\mu)}\,,\quad g_{22}=\frac{(1+\varepsilon_{1z}^0)^2}{2(1+\mu)}+\frac{\varepsilon_{1x}^0}{1-\mu^2}\,,\quad g_{23}=\frac{(1-\mu)(1+\varepsilon_{1z}^0)^2}{(1+\mu)(1-2\mu)}$$

附录 B

$$\Gamma_{1i}=(-g_{11}+g_{12}\alpha_{1i}^2)/(g_{13}\alpha_{1i})\,,\quad \Gamma_{2i}=E_1[(1-\mu)(1+\varepsilon_{1x}^0)+\mu(1+\varepsilon_{1z}^0)\Gamma_{1i}\alpha_{1i}]$$

$$\Gamma_{3i}=E_1[(1-\mu)(1+\varepsilon_{1z}^0)\Gamma_{1i}\alpha_{1i}+\mu(1+\varepsilon_{1x}^0)]\,,\quad \Gamma_{4i}=E[(1+\varepsilon_{1x}^0)\alpha_{1i}-(1+\varepsilon_{1z}^0)\Gamma_{1i}]/[2(1+\mu)]$$

$$E_1=E/[(1+\mu)(1-2\mu)]$$

附录 C

$$\xi_{1i}^A=\frac{\alpha_i(g_{12}\alpha_i^2+g_{12}\beta_i^2-g_{11})}{g_{13}(\alpha_i^2+\beta_i^2)}\,,\quad \zeta_{1i}^A=\frac{\beta_i(g_{12}\alpha_i\beta_i+g_{12}\beta_i^2+g_{11})}{g_{13}(\alpha_i^2+\beta_i^2)}\,,\quad \xi_{1i}^B=-\zeta_{1i}^A\,,\quad \zeta_{1i}^B=\xi_{1i}^A$$

$$\xi_{1i}^C=E_1[(1-\mu)(1+\varepsilon_{1x}^0)+\mu(1+\varepsilon_{1z}^0)(\alpha_i\xi_{1i}^A+\beta_i\xi_{1i}^B)]\,,\quad \zeta_{1i}^C=E_1[\mu(1+\varepsilon_{1z}^0)(\alpha_i\zeta_{1i}^A+\beta_i\zeta_{1i}^B)]\,,\quad \zeta_{1i}^D=\xi_{1i}^C\,,\quad \xi_{1i}^D=-\zeta_{1i}^C$$

$$\xi_{1i}^E=E_1[(1-\mu)(1+\varepsilon_{1z}^0)(\alpha_i\xi_{1i}^A+\beta_i\xi_{1i}^B)+\mu(1+\varepsilon_{1x}^0)]$$

$$\zeta_{1i}^E=E_1[(1-\mu)(1+\varepsilon_{1z}^0)(\alpha_i\zeta_{1i}^A+\beta_i\zeta_{1i}^B)]\,,\quad \zeta_{1i}^F=\xi_{1i}^E\,,\quad \xi_{1i}^F=-\zeta_{1i}^E$$

$$\xi_{1i}^G=G[(1+\varepsilon_{1x}^0)\alpha_i-(1+\varepsilon_{1z}^0)\xi_{1i}^A]\,,\quad \zeta_{1i}^G=G[(1+\varepsilon_{1x}^0)\beta_i-(1+\varepsilon_{1z}^0)\zeta_{1i}^A]\,,\quad \zeta_{1i}^H=\xi_{1i}^G\,,\quad \xi_{1i}^H=-\zeta_{1i}^G$$

附录 D

$$b_{11}=c_{11}(1+\varepsilon_0)^2+(c_{11}-c_{13}^2/c_{33})\varepsilon_0\,,\quad b_{12}=c_{44}(1+\varepsilon_0)^2\,,\quad b_{13}=(1+\varepsilon_0)(1+\varepsilon_{2z}^0)(c_{13}+c_{44})$$

$$b_{14}=(1+\varepsilon_0)(e_{15}+e_{31})\,,\quad b_{21}=b_{13}\,,\quad b_{22}=c_{44}(1+\varepsilon_{2z}^0)^2+\sigma_{2x}^0\,,\quad b_{23}=c_{33}(1+\varepsilon_{2z}^0)^2\,,\quad b_{24}=e_{15}(1+\varepsilon_{2z}^0)$$

$$b_{25}=e_{33}(1+\varepsilon_{2z}^0)\,,\quad b_{31}=e_{15}(1+\varepsilon_0)^2+e_{31}(1+\varepsilon_0)(1+\varepsilon_{2z}^0)+D_{2z}^0\,,\quad b_{32}=e_{15}(1+\varepsilon_0)(1+\varepsilon_{2z}^0)$$

$$b_{33}=e_{33}(1+\varepsilon_{2z}^0)^2+D_{2z}^0\,,\quad b_{34}=-\lambda_{11}(1+\varepsilon_0)\,,\quad b_{35}=-\lambda_{33}(1+\varepsilon_{2z}^0)$$

附录 E

$$\xi_{2i}^{(2)}=\frac{\begin{vmatrix} b_{11}-b_{12}(\alpha_{2i})^2 & -b_{14}\alpha_{2i} \\ -b_{21}\alpha_{2i} & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}{\begin{vmatrix} -b_{13}\alpha_{2i} & -b_{14}\alpha_{2i} \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}\,,\quad \zeta_{2i}^{(2)}=\frac{\begin{vmatrix} -b_{13}\alpha_{2i} & b_{11}-b_{12}(\alpha_{2i})^2 \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{21}\alpha_{2i} \end{vmatrix}}{\begin{vmatrix} -b_{13}\alpha_{2i} & -b_{14}\alpha_{2i} \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}$$

$$\Gamma_{1i}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)\xi_{2i}^{(2)}\alpha_{2i}+e_{31}\zeta_{2i}^{(2)}\alpha_{2i}\,,\quad \Gamma_{2i}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)\xi_{2i}^{(2)}\alpha_{2i}+e_{33}\zeta_{2i}^{(2)}\alpha_{2i}$$

$$\Gamma_{3i}^{(2)}=c_{44}(1+\varepsilon_0)\alpha_{2i}-c_{44}(1+\varepsilon_{2z}^0)\xi_{2i}^{(2)}-e_{15}\zeta_{2i}^{(2)}\,,\quad \Gamma_{4i}^{(2)}=e_{15}(1+\varepsilon_0)\alpha_{2i}-e_{15}(1+\varepsilon_{2z}^0)\xi_{2i}^{(2)}+\lambda_{11}\zeta_{2i}^{(2)}$$

$$\Gamma_{5i}^{(2)}=e_{31}(1+\varepsilon_0)+e_{33}(1+\varepsilon_{2z}^0)\xi_{2i}^{(2)}\alpha_{2i}-\lambda_{33}\zeta_{2i}^{(2)}\alpha_{2i}\quad (i=1\sim 6)$$

附录 F

$$\xi_{2i}^{(2)}=\frac{\begin{vmatrix} b_{11}-b_{12}(\alpha_{2i})^2 & -b_{14}\alpha_{2i} \\ -b_{21}\alpha_{2i} & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}{\begin{vmatrix} -b_{13}\alpha_{2i} & -b_{14}\alpha_{2i} \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}\,,\quad \zeta_{2i}^{(2)}=\frac{\begin{vmatrix} -b_{13}\alpha_{2i} & b_{11}-b_{12}(\alpha_{2i})^2 \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{21}\alpha_{2i} \end{vmatrix}}{\begin{vmatrix} -b_{13}\alpha_{2i} & -b_{14}\alpha_{2i} \\ -b_{22}+b_{23}(\alpha_{2i})^2 & -b_{24}+b_{25}(\alpha_{2i})^2 \end{vmatrix}}\quad (i=1,2)$$

$$\xi_{33}=\begin{vmatrix}\boldsymbol{\Theta}_{15} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|\,,\quad \xi_{34}=\begin{vmatrix}\boldsymbol{\Theta}_{16} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|$$

$$\xi_{43}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{15} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|\,,\quad \xi_{44}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{16} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|$$

$$\zeta_{33}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{15} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|\,,\quad \zeta_{34}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{16} & \boldsymbol{\Theta}_{14}\end{vmatrix}/|\boldsymbol{\Delta}_1|$$

$$\zeta_{43}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{15}\end{vmatrix}/|\boldsymbol{\Delta}_1|\,,\quad \zeta_{44}=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{16}\end{vmatrix}/|\boldsymbol{\Delta}_1|$$

$$\xi_{55}=\begin{vmatrix}\boldsymbol{\Theta}_{25} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|\,,\quad \xi_{56}=\begin{vmatrix}\boldsymbol{\Theta}_{26} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|$$

$$\xi_{65}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{25} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|\,,\quad \xi_{66}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{26} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|$$

$$\zeta_{55}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{25} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|\,,\quad \zeta_{56}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{26} & \boldsymbol{\Theta}_{24}\end{vmatrix}/|\boldsymbol{\Delta}_2|$$

$$\zeta_{65}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{25}\end{vmatrix}/|\boldsymbol{\Delta}_2|\,,\quad \zeta_{66}=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{26}\end{vmatrix}/|\boldsymbol{\Delta}_2|$$

$$|\boldsymbol{\Delta}_1|=\begin{vmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{13} & \boldsymbol{\Theta}_{14}\end{vmatrix}\,,\quad |\boldsymbol{\Delta}_2|=\begin{vmatrix}\boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{23} & \boldsymbol{\Theta}_{24}\end{vmatrix}$$

$$\boldsymbol{\Theta}_{11}=\left\{-b_{13}\alpha_2 \quad b_{13}\beta_2 \quad -b_{22}+b_{23}\alpha_2^2-b_{23}\beta_2^2 \quad -2b_{23}\alpha_2\beta_2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{12}=\left\{-b_{13}\beta_2 \quad -b_{13}\alpha_2 \quad 2b_{23}\alpha_2\beta_2 \quad -b_{22}+b_{23}\alpha_2^2-b_{23}\beta_2^2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{13}=\left\{-b_{14}\alpha_2 \quad b_{14}\beta_2 \quad -b_{24}+b_{25}\alpha_2^2-b_{25}\beta_2^2 \quad -2b_{25}\alpha_2\beta_2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{14}=\left\{-b_{14}\beta_2 \quad -b_{14}\alpha_2 \quad 2b_{25}\alpha_2\beta_2 \quad -b_{24}+b_{25}\alpha_2^2-b_{25}\beta_2^2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{15}=\left\{b_{11}-b_{12}\alpha_2^2+b_{12}\beta_2^2 \quad 2b_{12}\alpha_2\beta_2 \quad -b_{21}\alpha_2 \quad b_{21}\beta_2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{16}=\left\{-2b_{12}\alpha_2\beta_2 \quad b_{11}-b_{12}\alpha_2^2+b_{12}\beta_2^2 \quad -b_{21}\beta_2 \quad -b_{21}\alpha_2\right\}^{\rm T}$$

$$\boldsymbol{\Theta}_{21}=\left\{\,b_{13}\alpha_2\quad b_{13}\beta_2\quad -b_{22}+b_{23}\alpha_2^2-b_{23}\beta_2^2\quad 2b_{23}\alpha_2\beta_2\,\right\}^{\mathrm{T}}$$

$$\boldsymbol{\Theta}_{22}=\left\{\,-b_{13}\beta_2\quad b_{13}\alpha_2\quad -2b_{23}\alpha_2\beta_2\quad -b_{22}+b_{23}\alpha_2^2-b_{23}\beta_2^2\,\right\}^{\mathrm{T}}$$

$$\boldsymbol{\Theta}_{23}=\left\{\,b_{14}\alpha_2\quad b_{14}\beta_2\quad -b_{24}+b_{25}\alpha_2^2-b_{25}\beta_2^2\quad 2b_{25}\alpha_2\beta_2\,\right\}^{\mathrm{T}}$$

$$\boldsymbol{\Theta}_{24}=\left\{\,-b_{14}\beta_2\quad b_{14}\alpha_2\quad -2b_{25}\alpha_2\beta_2\quad -b_{24}+b_{25}\alpha_2^2-b_{25}\beta_2^2\,\right\}^{\mathrm{T}}$$

$$\boldsymbol{\Theta}_{25}=\left\{\,b_{11}-b_{12}\alpha_2^2+b_{12}\beta_2^2\quad -2b_{12}\alpha_2\beta_2\quad b_{21}\alpha_2\quad b_{21}\beta_2\,\right\}^{\mathrm{T}}$$

$$\boldsymbol{\Theta}_{26}=\left\{\,2b_{12}\alpha_2\beta_2\quad b_{11}-b_{12}\alpha_2^2+b_{12}\beta_2^2\quad -b_{21}\beta_2\quad b_{21}\alpha_2\,\right\}^{\mathrm{T}}$$

附录 G

$\Gamma_{11}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)\xi_{21}^{(2)}\alpha_{21}+e_{31}\zeta_{21}^{(2)}\alpha_{21}$， $\Gamma_{12}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)\xi_{22}^{(2)}\alpha_{22}+e_{31}\zeta_{22}^{(2)}\alpha_{22}$

$\Gamma_{13}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{33}+\beta_2\xi_{43})+e_{31}(\alpha_2\zeta_{33}+\beta_2\zeta_{43})$， $\Gamma_{14}^{(2)}=c_{13}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{34}+\beta_2\xi_{44})+e_{31}(\alpha_2\zeta_{34}+\beta_2\zeta_{44})$

$\Gamma_{15}^{(2)}=c_{13}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{43}-\beta_2\xi_{33})+e_{31}(\alpha_2\zeta_{43}-\beta_2\zeta_{33})$， $\Gamma_{16}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{44}-\beta_2\xi_{34})+e_{31}(\alpha_2\zeta_{44}-\beta_2\zeta_{34})$

$\Gamma_{17}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{55}+\beta_2\xi_{65})+e_{31}(-\alpha_2\zeta_{55}+\beta_2\zeta_{65})$

$\Gamma_{18}^{(2)}=c_{13}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{56}+\beta_2\xi_{66})+e_{31}(-\alpha_2\zeta_{56}+\beta_2\zeta_{66})$， $\Gamma_{19}^{(2)}=c_{13}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{65}-\beta_2\xi_{55})+e_{31}(-\alpha_2\zeta_{65}-\beta_2\zeta_{55})$

$\Gamma_{1(10)}^{(2)}=c_{11}(1+\varepsilon_0)+c_{13}(1-c_{13}\varepsilon_0/c_{33})(-\alpha_2\xi_{66}-\beta_2\xi_{56})+e_{31}(-\alpha_2\zeta_{66}-\beta_2\zeta_{56})$

$\Gamma_{21}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)\xi_{21}^{(2)}\alpha_{21}+e_{33}\zeta_{21}^{(2)}\alpha_{21}$， $\Gamma_{22}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)\xi_{22}^{(2)}\alpha_{22}+e_{33}\zeta_{22}^{(2)}\alpha_{22}$

$\Gamma_{23}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{33}+\beta_2\xi_{43})+e_{33}(\alpha_2\zeta_{33}+\beta_2\zeta_{43})$， $\Gamma_{24}^{(2)}=c_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{34}+\beta_2\xi_{44})+e_{33}(\alpha_2\zeta_{34}+\beta_2\zeta_{44})$

$\Gamma_{25}^{(2)}=c_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{43}-\beta_2\xi_{33})+e_{33}(\alpha_2\zeta_{43}-\beta_2\zeta_{33})$， $\Gamma_{26}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{44}-\beta_2\xi_{34})+e_{33}(\alpha_2\zeta_{44}-\beta_2\zeta_{34})$

$\Gamma_{27}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{55}+\beta_2\xi_{65})+e_{33}(-\alpha_2\zeta_{55}+\beta_2\zeta_{65})$

$\Gamma_{28}^{(2)}=c_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{56}+\beta_2\xi_{66})+e_{33}(-\alpha_2\zeta_{56}+\beta_2\zeta_{66})$， $\Gamma_{29}^{(2)}=c_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{65}-\beta_2\xi_{55})+e_{33}(-\alpha_2\zeta_{65}-\beta_2\zeta_{55})$

$\Gamma_{2(10)}^{(2)}=c_{13}(1+\varepsilon_0)+c_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{66}-\beta_2\xi_{56})+e_{33}(-\alpha_2\zeta_{66}-\beta_2\zeta_{56})$

$\Gamma_{31}^{(2)}=c_{44}(1+\varepsilon_0)\alpha_{21}-c_{44}(1+\varepsilon_{2z}^0)\xi_{21}^{(2)}-e_{15}\zeta_{21}^{(2)}$， $\Gamma_{32}^{(2)}=c_{44}(1+\varepsilon_0)\alpha_{22}-c_{44}(1+\varepsilon_{2z}^0)\xi_{22}^{(2)}-e_{15}\zeta_{22}^{(2)}$

$\Gamma_{33}^{(2)}=c_{44}(1+\varepsilon_0)\alpha_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{33}-e_{15}\zeta_{33}$， $\Gamma_{34}^{(2)}=c_{44}(1+\varepsilon_0)\beta_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{34}-e_{15}\zeta_{34}$

$\Gamma_{35}^{(2)}=-c_{44}(1+\varepsilon_0)\beta_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{43}-e_{15}\zeta_{43}$， $\Gamma_{36}^{(2)}=c_{44}(1+\varepsilon_0)\alpha_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{44}-e_{15}\zeta_{44}$

$\Gamma_{37}^{(2)}=-c_{44}(1+\varepsilon_0)\alpha_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{55}-e_{15}\zeta_{55}$， $\Gamma_{38}^{(2)}=c_{44}(1+\varepsilon_0)\beta_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{56}-e_{15}\zeta_{56}$

$\Gamma_{39}^{(2)}=-c_{44}(1+\varepsilon_0)\beta_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{65}-e_{15}\zeta_{65}$， $\Gamma_{3(10)}^{(2)}=-c_{44}(1+\varepsilon_0)\alpha_2-c_{44}(1+\varepsilon_{2z}^0)\xi_{66}-e_{15}\zeta_{66}$

$\Gamma_{41}^{(2)}=e_{15}(1+\varepsilon_0)\alpha_{21}-e_{15}(1+\varepsilon_{2z}^0)\xi_{21}^{(2)}+\lambda_{11}\zeta_{21}^{(2)}$， $\Gamma_{42}^{(2)}=e_{15}(1+\varepsilon_0)\alpha_{22}-e_{15}(1+\varepsilon_{2z}^0)\xi_{22}^{(2)}+\lambda_{11}\zeta_{22}^{(2)}$

$\Gamma_{43}^{(2)}=e_{15}(1+\varepsilon_0)\alpha_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{33}+\lambda_{11}\zeta_{33}$， $\Gamma_{44}^{(2)}=e_{15}(1+\varepsilon_0)\beta_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{34}+\lambda_{11}\zeta_{34}$

$\Gamma_{45}^{(2)}=-e_{15}(1+\varepsilon_0)\beta_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{43}+\lambda_{11}\zeta_{43}$， $\Gamma_{46}^{(2)}=e_{15}(1+\varepsilon_0)\alpha_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{44}+\lambda_{11}\zeta_{44}$

$\Gamma_{47}^{(2)}=-e_{15}(1+\varepsilon_0)\alpha_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{55}+\lambda_{11}\zeta_{55}$， $\Gamma_{48}^{(2)}=e_{15}(1+\varepsilon_0)\beta_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{56}+\lambda_{11}\zeta_{56}$

$\Gamma_{49}^{(2)}=-e_{15}(1+\varepsilon_0)\beta_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{65}+\lambda_{11}\zeta_{65}$， $\Gamma_{4(10)}^{(2)}=-e_{15}(1+\varepsilon_0)\alpha_2-e_{15}(1+\varepsilon_{2z}^0)\xi_{66}+\lambda_{11}\zeta_{66}$

$\Gamma_{51}^{(2)}=e_{31}(1+\varepsilon_0)+e_{33}(1+\varepsilon_{2z}^0)\xi_{21}^{(2)}\alpha_{21}-\lambda_{33}\zeta_{21}^{(2)}\alpha_{21}$， $\Gamma_{52}^{(2)}=e_{31}(1+\varepsilon_0)+e_{33}(1+\varepsilon_{2z}^0)\xi_{22}^{(2)}\alpha_{22}-\lambda_{33}\zeta_{22}^{(2)}\alpha_{22}$

$\Gamma_{53}^{(2)}=e_{31}(1+\varepsilon_0)+e_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{33}+\beta_2\xi_{43})-\lambda_{33}(\alpha_2\zeta_{33}+\beta_2\zeta_{43})$， $\Gamma_{54}^{(2)}=e_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{34}+\beta_2\xi_{44})-\lambda_{33}(\alpha_2\zeta_{34}+\beta_2\zeta_{44})$

$\Gamma_{55}^{(2)}=e_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{43}-\beta_2\xi_{33})-\lambda_{33}(\alpha_2\zeta_{43}-\beta_2\zeta_{33})$， $\Gamma_{56}^{(2)}=e_{31}(1+\varepsilon_0)+e_{33}(1+\varepsilon_{2z}^0)(\alpha_2\xi_{44}-\beta_2\xi_{34})-\lambda_{33}(\alpha_2\zeta_{44}-\beta_2\zeta_{34})$

$$\Gamma_{57}^{(2)} = e_{31}(1+\varepsilon_0) + e_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{55}+\beta_2\xi_{65}) - \lambda_{33}(-\alpha_2\zeta_{55}+\beta_2\zeta_{65})$$

$$\Gamma_{58}^{(2)} = e_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{56}+\beta_2\xi_{66}) - \lambda_{33}(-\alpha_2\zeta_{56}+\beta_2\zeta_{66}), \quad \Gamma_{59}^{(2)} = e_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{65}-\beta_2\xi_{55}) - \lambda_{33}(-\alpha_2\zeta_{65}-\beta_2\zeta_{55})$$

$$\Gamma_{5(10)}^{(2)} = e_{31}(1+\varepsilon_0) + e_{33}(1+\varepsilon_{2z}^0)(-\alpha_2\xi_{66}-\beta_2\xi_{56}) - \lambda_{33}(-\alpha_2\zeta_{66}-\beta_2\zeta_{56})$$

BUCKLING ANALYSIS OF THE ELECTRODE DELAMINATION ON THE PIEZOELECTRIC SUBSTRATE [1)]

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Abstract The paper investigated the electrode delamination buckling of the layered system containing a through-the-width delamination between the metallic electrode and the half-space piezoelectric substrate based on the finite deformation theory of elasticity and the biasing field theory of the electroelastic body. The layered system in the plane strain problem is subjected to the compressive strain-load parallel to the free surface. Meanwhile, the theoretical model is reduced to the second kind Cauchy-type singular integral equations by means of the Fourier integral transform, the boundary conditions and the interfacial continuous conditions. The singular integral equations are solved numerically by utilizing Gauss-Chebyshev integral formulae. As an example, the layered system of the metallic electrode Pt and piezoelectric substrate PZT-4 is considered. Numerical results for the critical strains of buckling and the corresponding delamination buckling shapes are presented for, respectively, various ratios of the delamination length to thickness and the effect of electromechanical coupling in the piezoelectric substrate. The curves of the singular oscillating factors in the delamination tip with respect to the ratios of delamination length to thickness are also given.

Key words electrode delamination, buckling, Fourier integral transform, singular integral functions, critical strains

Received 4 December 2007, revised 26 August 2008.
1) The project supported by the Ph.D. Programs Foundation of Ministry of Education of China(20050247004).
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