

功能梯度压电材料圆板的简化理论与解析解¹⁾

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摘要 提出了功能梯度圆板在轴对称载荷作用下的简化理论与解析解. 引入了板理论的若干假设 (Kirchhoff 假设的一部分, Reissner-Mindlin 假设和文中提出的假设), 并假设材料常数在板厚方向按指数规律变化. 推导了板的周边固定或简支同时又接地情况下中性层法线转角的解和用 Fourier-Bessel 级数表示的电势解. 这个解在形式上比精确解简单得多, 进行数值计算时也相当方便与快捷. 该文给出了板的周边固定、接地情况下的计算结果并进行了讨论, 对于理论和方法的正确性作了验证.

关键词 功能梯度压电材料, 轴对称, 圆板, 简化理论, 解析解

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引 言

功能梯度材料 (FGM) 一般由两种或两种以上的材料复合而成, 各组分材料的体积含量在空间位置上连续变化, 因而其宏观材料特性表现出梯度 (逐渐变化) 的性质. 由于这类材料的力学和热学参数没有突变, 因而大大缓解了应力集中. 功能梯度材料是一种设计思想新颖、性能良好的新材料, 会有很好的发展前景.

压电材料以其特有的机电耦合特性, 被广泛应用于各种工程领域. 功能梯度压电材料, 由于其物性参数是空间坐标的函数, 而且存在着力 - 电 - 热的耦合作用, 使得该种材料的力学分析变得相当复杂. 从应用力学的观点来看, 功能梯度压电材料不仅是各向异性的而且是非均匀的, 这些性质对材料的力学和电学性能有很大影响. 深入研究该种材料的力学行为, 对于功能梯度材料的设计、制造以及工程应用具有十分重要的实际意义.

国内外对功能梯度材料和压电复合材料的力学研究已有大量的成果. 吴瑞安和仲政等^[1]对功能梯度压电材料平板进行了力电耦合结构分析; 仲政和尚尔涛^[2]对功能梯度压电材料矩形板进行了热力电耦合的静力分析; 陈伟球和丁皓江^[3]研究了功能梯度压电材料矩形板的自由振动问题; Tanigawa 等^[4]用层合模型研究了平板的一维非定常热应力问

题; Noda 和 Tsuji^[5]研究了沿平板的厚度方向的一维导热问题; Takashi 和 Noda^[6]研究了功能复合材料板的热应力; Yang 等^[7]研究了力 - 电 - 热耦合作用下的功能梯度层合板的振动问题. Lucia 和 Paolo^[8]用有限元研究了 Reissner-Mindlin 功能梯度板的问题; 朱昊文等^[9]得到了功能梯度压电材料板的有限元解; 此外与本文比较接近的工作还有丁皓江等^[10,11]的轴对称层合横观各向同性压电材料圆板精确解的研究; 以及张晓日和仲政^[12]关于功能梯度压电圆板自由振动精确解的研究等.

目前对于功能梯度压电材料板的研究大多集中在某些特定边界条件下 (如四边简支、接地) 的矩形板问题的求解, 但对于应用广泛的功能梯度压电圆板结构的研究相对较少, 甚至对于一般圆板问题的三维分析工作也较少, 而且求解过程比较复杂. 本文通过板问题的 Reissner-Mindlin 假设和其它适当的假设, 给出简化理论下的形式较简单的解析解.

1 圆板问题的基本方程

设极化方向为 z 轴的正向, 材料是横观各向同性的, 可得功能梯度压电材料圆板轴对称问题的基本方程为:

应力平衡方程

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$$\left. \begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \right\} \quad (1)$$

Maxwell 电位移守恒方程

$$\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0 \quad (2)$$

几何关系

$$\left. \begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} \\ \varepsilon_z &= \frac{\partial u_z}{\partial z}, \quad \gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \end{aligned} \right\} \quad (3)$$

电场与电势的关系

$$E_r = -\frac{\partial \phi}{\partial r}, \quad E_z = -\frac{\partial \phi}{\partial z} \quad (4)$$

本构方程

$$\left. \begin{aligned} \sigma_r &= C_{11}\varepsilon_r + C_{12}\varepsilon_\theta + C_{13}\varepsilon_z - e_{31}E_z \\ \sigma_\theta &= C_{12}\varepsilon_r + C_{11}\varepsilon_\theta + C_{13}\varepsilon_z - e_{31}E_z \\ \sigma_z &= C_{13}\varepsilon_r + C_{13}\varepsilon_\theta + C_{33}\varepsilon_z - e_{33}E_z \\ \tau_{rz} &= C_{44}\gamma_{rz} - e_{15}E_r \\ D_r &= e_{15}\gamma_{rz} + \lambda_{11}E_r \\ D_z &= e_{31}(\varepsilon_r + \varepsilon_\theta) + e_{33}\varepsilon_z + \lambda_{33}E_z \end{aligned} \right\} \quad (5)$$

式中 $\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$ 为应力; $\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{rz}$ 为应变; u_r, u_z 为位移; D_r, D_z 为电位移; E_r, E_z 为电场; ϕ 为电势; $C_{11}, C_{12}, C_{13}, C_{33}, C_{44}$ 为弹性系数; e_{31}, e_{33}, e_{15} 为压电系数; $\lambda_{11}, \lambda_{33}$ 为介电系数. 对于功能梯度材料, 我们设所有的材料参数可以用同一个梯度函数表示, 即

$$K = K^0 f(z) \quad (6)$$

式中 K 为某材料参数, K^0 为 $z = 0$ 处该参数的值, 函数 $f(z)$ 要求 $f(0) = 1$.

2 圆板的简化理论

对于小挠度的薄板, 略去一些次要的因素而引用一些能够简化理论的假设, 所造成的误差不会对结果有多大的影响. 我们这里引入 3 条假设:

(1) 变形前的中性面法线在变形后仍保持为直线, 但不一定与中性面垂直. 这就是著名的 Reissner-Mindlin 假设.

(2) 与面内应力 $\sigma_r, \sigma_\theta, \tau_{r\theta}$ 相比, 另外 3 个应力分量 $\sigma_z, \tau_{rz}, \tau_{z\theta}$ 很小, 在计算应变时可以忽略不计. 这就是著名的 Kirchhoff 假设.

(3) 与面内应力 $\sigma_r, \sigma_\theta, \tau_{r\theta}$ 相比, 另外 3 个应力分量 $\sigma_z, \tau_{rz}, \tau_{z\theta}$ 很小, 在计算电位移时也可忽略不计. 这是本文提出的假设.

边界条件如下:

(1) 板的上下表面的力载荷已知, 电势或电位移分布已知.

(2) 板的周边接地, 即电势为 0.

(3) 板的周边简支, 即挠度和弯矩为 0; 或

(4) 板的周边固支, 即挠度和转角为 0.

由于材料性质沿板厚方向变化, 板中面一般不是中性面. 可以用等效截面法求出中性面的位置. 把 $z = 0$ 设置在中性层上, 就有

$$\int_{-h_1}^{h_2} z f(z) dz = 0 \quad (7)$$

式中 h 为板厚; h_1, h_2 分别为中性层到板的下表面和上表面的距离.

根据假设 (1), 板内的径向位移为

$$u_r = u_0 + \beta z \quad (8)$$

式中 u_0 为板中性层内的位移, β 为中性层法线绕圆板环向线的转角.

3 β 解的推导

根据上节第 (2) 条假定, 式 (5) 的第 3 式变为

$$\varepsilon_z = -\frac{C_{13}}{C_{33}}(\varepsilon_r + \varepsilon_\theta) + \frac{e_{33}}{C_{33}}E_z$$

代入式 (5) 中的第 1, 2 式得

$$\sigma_r = \bar{C}_{11}\varepsilon_r + \bar{C}_{12}\varepsilon_\theta - \bar{e}_{31}E_z \quad (9)$$

$$\sigma_\theta = \bar{C}_{12}\varepsilon_r + \bar{C}_{11}\varepsilon_\theta - \bar{e}_{31}E_z \quad (10)$$

$$D_z = \bar{e}_{31}(\varepsilon_r + \varepsilon_\theta) + \bar{\lambda}_{33}E_z \quad (11)$$

式中

$$\bar{C}_{11} = C_{11} - \frac{C_{13}^2}{C_{33}}, \quad \bar{C}_{12} = C_{12} - \frac{C_{13}^2}{C_{33}}$$

$$\bar{e}_{31} = e_{31} - \frac{e_{33}C_{13}}{C_{33}}, \quad \bar{\lambda}_{33} = \lambda_{33} + \frac{e_{33}^2}{C_{33}}$$

将方程 (9), (10) 代入方程 (1) 的第 1 式, 并利用式

(3) 和式 (8) 得

$$\begin{aligned} \frac{\partial \tau_{zr}}{\partial z} = & -\bar{C}_{11}(u_{0,rr} + z\beta_{,rr}) - \\ & \bar{C}_{12}\left(-\frac{u_0}{r^2} + \frac{1}{r}u_{0,r} - \frac{\beta}{r^2}z + \frac{1}{r}\beta_{,r}z\right) - \\ & \bar{e}_{31}\phi_{,zr} - \frac{1}{r}\left[(\bar{C}_{11} - \bar{C}_{12})(u_{0,r} + \beta_{,r}z) + \right. \\ & \left. (\bar{C}_{12} - \bar{C}_{11})\left(\frac{u_0}{r} + \frac{\beta z}{r}\right)\right] \end{aligned} \quad (12)$$

记

$$F(z) = \int_0^z f(z)dz, \quad F_1(z) = \int_0^z zf(z)dz \quad (13)$$

方程 (12) 对 z 积分后得

$$\begin{aligned} \tau_{zr} = \tau_{zr}|_{z=0} - \bar{C}_{11}^0 u_{0,rr}F(z) - \bar{C}_{11}^0 \beta_{,rr}F_1(z) + \\ \bar{C}_{12}^0 \frac{u_0}{r^2}F(z) - \bar{C}_{12}^0 \frac{u_{0,r}}{r}F(z) + \frac{\bar{C}_{12}^0 \beta}{r^2}F_1(z) - \\ \bar{C}_{12}^0 \frac{1}{r}\beta_{,r}F_1(z) - \bar{e}_{31}^0 \frac{\partial}{\partial r} \int_0^z \phi_{,z}f(z)dz + \\ \frac{\bar{C}_{11}^0 - \bar{C}_{12}^0}{r} \left[\frac{u_0}{r}F(z) + \frac{\beta}{r}F_1(z) - \right. \\ \left. u_{0,r}F(z) - \beta_{,r}F_1(z) \right] \end{aligned} \quad (14)$$

利用条件 $\tau_{zr}|_{z=-h_1} = 0$, $\tau_{zr}|_{z=h_2} = 0$, 并注意到

$$\begin{aligned} F_1(h_2) - F_1(-h_1) = \int_0^{h_2} zf(z)dz - \int_0^{-h_1} zf(z)dz = \\ \int_{-h_1}^{h_2} zf(z)dz = 0 \end{aligned}$$

可得中性层处的剪应力为

$$\begin{aligned} \tau_{zr}|_{z=0} = -\bar{e}_{31}^0 \frac{1}{G} F(-h_1) \frac{d}{dr} \int_{-h_1}^{h_2} \phi_{,z}f(z)dz + \\ \bar{C}_{11}^0 \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\beta) \right] F_1(z) + \bar{e}_{31}^0 \frac{d}{dr} \int_0^{-h_1} \phi_{,z}f(z)dz \end{aligned} \quad (15)$$

式中

$$G = F(h_2) - F(-h_1) = \int_{-h_1}^{h_2} f(z)dz \quad (16)$$

代入式 (14), 化简后得

$$\begin{aligned} \tau_{zr} = \bar{C}_{11}^0 \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\beta) \right] [F_1(-h_1) - F_1(z)] + \\ \bar{e}_{31}^0 \left\{ \frac{1}{G} [F(z) - F(-h_1)] \frac{d}{dr} \int_{-h_1}^{h_2} \phi_{,z}f(z)dz - \right. \\ \left. \frac{\partial}{\partial r} \int_{-h_1}^z \phi_{,z}f(z)dz \right\} \end{aligned} \quad (17)$$

将式 (17) 代入方程 (1) 中的第 2 式, 积分后得

$$\begin{aligned} \sigma_z = \sigma_z|_{z=0} - \frac{1}{r} \frac{d}{dr} \left\{ \bar{C}_{11}^0 r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\beta) \right] \cdot \right. \\ \left. [zF_1(-h_1) - \int_0^z F_1(z)dz] + \right. \\ \left. \bar{e}_{31}^0 r \left[\frac{1}{G} \left(\int_0^z F(z)dz - zF(-h_1) \right) \cdot \right. \right. \\ \left. \frac{d}{dr} \int_{-h_1}^{h_2} \phi_{,z}f(z)dz - \right. \\ \left. \frac{d}{dr} \int_0^z \left(\int_0^z \phi_{,z_1}f(z_1)dz_1 \right) dz + \right. \\ \left. \left. \frac{d}{dr} z \int_0^{-h_1} \phi_{,z}f(z)dz \right] \right\} \end{aligned} \quad (18)$$

上下板面载荷之差 $q(r) = \sigma_z(r, h_2) - \sigma_z(r, -h_1)$, 将上式代入后得

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left\{ \bar{C}_{11}^0 r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\beta) \right] A + \right. \\ \left. \bar{e}_{31}^0 r \left[\frac{B}{G} \frac{d}{dr} \int_{-h_1}^{h_2} \phi_{,z}f(z)dz - \right. \right. \\ \left. \left. \frac{d}{dr} \int_{-h_1}^{h_2} \phi_{,z}f(z)(h_2 - z)dz \right] \right\} = -q(r) \end{aligned} \quad (19)$$

式中

$$\begin{aligned} A = hF_1(-h_1) - \int_{-h_1}^{h_2} F_1(z)dz \\ B = \int_{-h_1}^{h_2} F(z)dz - hF(-h_1) \end{aligned} \quad (20)$$

方程 (19) 对 r 积分两次得

$$\begin{aligned} \bar{C}_{11}^0 \left[\frac{1}{r} \frac{d}{dr} (r\beta) \right] A + \bar{e}_{31}^0 \left[\frac{B}{G} \int_{-h_1}^{h_2} \phi_{,z}f(z)dz - \right. \\ \left. \int_{-h_1}^{h_2} \phi_{,z}f(z)(h_2 - z)dz \right] = \\ - \int_0^r \frac{1}{r} \left(\int_0^r rq(r)dr \right) dr + D_1 \ln r + D_2 \end{aligned} \quad (21)$$

当 $r = 0$ 时, M_r 有限. 由此条件可确定 $D_1 = 0$. 将此结果代入式 (21), 再对 r 积分得

$$\begin{aligned} \beta = -\frac{\bar{e}_{31}^0}{\bar{C}_{11}^0 A r} \left[\frac{B}{G} \int_{-h_1}^{h_2} \left(\int_0^r r\phi_{,z}dr \right) f(z)dz - \right. \\ \left. \int_{-h_1}^{h_2} \left(\int_0^r r\phi_{,z}dr \right) f(z)(h_2 - z)dz \right] - \\ \frac{1}{\bar{C}_{11}^0 A r} \left[\int_0^r r \left(\int_0^r \frac{1}{r} \left(\int_0^r rq(r)dr \right) dr \right) dr - \right. \\ \left. D_2 \frac{r^2}{2} \right] + D_3 \end{aligned} \quad (22)$$

由条件 $r = 0$ 时 $\beta = 0$ 可得 $D_3 = 0$. 积分常数 D_2 可用板周边的支承条件确定:

(1) 板周边固支时, 有 $r = a$ 时 $\beta = 0, \phi = 0$, 利用式 (22) 得

$$D_2 = Q_1 + D'_2 \quad (23)$$

式中

$$Q_1 = \frac{2}{a^2} \left\{ \int_0^a r \left[\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right] dr \right\} \quad (24)$$

$$D'_2 = \frac{2\bar{e}_{31}^0}{a^2} \left[\frac{B}{G} \int_{-h_1}^{h_2} \left(\int_0^a r \phi_{,z} dr \right) f(z) dz - \int_{-h_1}^{h_2} \left(\int_0^a r \phi_{,z} dr \right) f(z) (h_2 - z) dz \right] \quad (25)$$

(2) 板周边简支时, 有 $r = a$ 时 $\phi = 0, M_r = 0$. 利用式 (3), (4) 和 (5) 的第 1 式得

$$M_r = \int_{-h_1}^{h_2} z \sigma_r dz = \int_{-h_1}^{h_2} z \left[\bar{C}_{11}^0 f(z) z \beta_{,r} + \bar{C}_{12}^0 f(z) z \frac{\beta}{r} + \bar{e}_{31}^0 f(z) \phi_{,z} \right] dz = 0 \quad (26)$$

将式 (22) 代入再化简, 可得

$$D_2 = Q_2 + D'_2 \quad (27)$$

式中

$$Q_2 = \frac{2\bar{C}_{11}^0}{\bar{C}_{11}^0 + \bar{C}_{12}^0} \int_0^a \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr + \frac{2(\bar{C}_{12}^0 - \bar{C}_{11}^0)}{\bar{C}_{11}^0 + \bar{C}_{12}^0} \frac{1}{a^2} \int_0^a r \left[\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right] dr \quad (28)$$

$$D'_2 = \frac{2(\bar{C}_{12}^0 - \bar{C}_{11}^0)}{\bar{C}_{11}^0 + \bar{C}_{12}^0} \frac{\bar{e}_{31}^0}{a^2} \int_0^a r \left[\left(\frac{B}{G} - h_2 \right) \int_{-h_1}^{h_2} \phi_{,z} f(z) dz + \int_{-h_1}^{h_2} \phi_{,z} f(z) z dz \right] dr \quad (29)$$

4 关于 ϕ 的微分方程及求解

将式 (4) 代入式 (5) 中的第 5 式, 再根据假设 (3), 略去 γ_{rz} 对电位移的影响, 得到

$$D_r = -\lambda_{11} \phi_{,r} \quad (30)$$

将式 (3) 代入式 (11) 再利用式 (8), 可得

$$D_z = \bar{e}_{31} \left[\frac{1}{r} \frac{d}{dr} (ru_0) + \frac{z}{r} \frac{d}{dr} (r\beta) \right] - \bar{\lambda}_{33} \phi_{,z} \quad (31)$$

设电势的解为

$$\phi = \sum_{i=0}^{\infty} Z_i(z) J_0 \left(\frac{\sqrt{\lambda_i}}{a} r \right) \quad (32)$$

其中 J_0 为 0 阶贝塞尔函数, $\sqrt{\lambda_i}$ 为贝塞尔函数的 0 点. 再引入无量纲变量 $\zeta = z/h, \rho = r/a$, 则有

$$\phi = \sum_{i=0}^{\infty} Z_i(\zeta) J_0(\sqrt{\lambda_i} \rho) \quad (33)$$

将式 (33) 代入式 (30) 和 (31), 再代入方程 (2), 得

$$\begin{aligned} & \sum_{i=0}^{\infty} J_0(\sqrt{\lambda_i} \rho) \left[\frac{\bar{\lambda}_{11}^0}{a^2} \lambda_i Z_i(\zeta) - \frac{\alpha \bar{\lambda}_{33}^0}{h^2} Z_i'(\zeta) - \frac{\bar{\lambda}_{33}^0}{h^2} Z_i''(\zeta) \right] = \\ & \bar{e}_{31}^0 (1 + \alpha \zeta) \left\{ \frac{\bar{e}_{31}^0}{\bar{C}_{11}^0 A} \sum_{i=0}^{\infty} \left[\left(\frac{B}{G} - h_2 \right) \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) \cdot \right. \right. \\ & \left. \left. e^{\alpha \zeta} d\zeta + h \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) e^{\alpha \zeta} \zeta d\zeta \right] J_0(\sqrt{\lambda_i} \rho) + \right. \\ & \left. \frac{1}{\bar{C}_{11}^0 A} \left[a^2 \int_0^{\rho} \frac{1}{\rho} \left(\int_0^{\rho} \rho q(\rho) d\rho \right) d\rho - D_2 \right] \right\} \quad (34) \end{aligned}$$

在以上的推导中略去 u_0 的影响, 并设 $f(z) = e^{\alpha \zeta}$. 再将上式最后一项展成 Bessel 级数, 即

$$\begin{aligned} & \frac{\bar{e}_{31}^0}{\bar{C}_{11}^0 A} \left[a^2 \int_0^{\rho} \frac{1}{\rho} \left(\int_0^{\rho} \rho q(\rho) d\rho \right) d\rho - D_2 \right] = \\ & \sum_{i=0}^{\infty} Y_i J_0(\sqrt{\lambda_i} \rho) \quad (35) \end{aligned}$$

代入式 (34), 比较级数各项的系数得

$$\begin{aligned} & \frac{\bar{\lambda}_{11}^0}{a^2} \lambda_i Z_i(\zeta) - \frac{\alpha \bar{\lambda}_{33}^0}{h^2} Z_i'(\zeta) - \frac{\bar{\lambda}_{33}^0}{h^2} Z_i''(\zeta) = \\ & (\alpha \zeta + 1) \left\{ \frac{(\bar{e}_{31}^0)^2}{\bar{C}_{11}^0 A} \left[\left(\frac{B}{G} - h_2 \right) \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) e^{\alpha \zeta} d\zeta + \right. \right. \\ & \left. \left. h \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) e^{\alpha \zeta} \zeta d\zeta \right] + Y_i \right\} \quad (36) \end{aligned}$$

上式用某个电势值 Φ 除, 并将 $\frac{Z_i(\zeta)}{\Phi}$ 仍表示为 $Z_i(\zeta)$, 两边再乘以 $\frac{a^2}{\bar{\lambda}_{11}^0}$, 可得关于 Fourier-Bessel 级数系数的无量纲微分方程

$$\lambda_i Z_i(\zeta) - \kappa [\alpha Z_i'(\zeta) + Z_i''(\zeta)] = s_i + t_i \zeta \quad (37)$$

式中

$$\begin{aligned} s_i = & \frac{(\bar{e}_{31}^0)^2 a^2}{\bar{C}_{11}^0 \bar{\lambda}_{11}^0 A} \left[\left(\frac{B}{G} - h_2 \right) \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) e^{\alpha \zeta} d\zeta + \right. \\ & \left. h \int_{-\zeta_1}^{\zeta_2} Z_i'(\zeta) e^{\alpha \zeta} \zeta d\zeta \right] + \bar{Y}_i, \quad t_i = \alpha s_i \quad (38) \end{aligned}$$

$$\kappa = \frac{\bar{\lambda}_{33}^0 a^2}{\bar{\lambda}_{11}^0 h^2}, \quad \bar{Y}_i = \frac{a^2}{\bar{\lambda}_{11}^0 \Phi} Y_i \quad (39)$$

方程 (37) 的解为

$$Z_i(\zeta) = M_i e^{\mu_i \zeta} + N_i e^{\nu_i \zeta} + S_i + T_i \zeta \quad (40)$$

式中 μ_i 和 ν_i 为式 (37) 对应齐次方程的两个根, 为

$$\left. \begin{matrix} \mu_i \\ \nu_i \end{matrix} \right\} = \frac{-\alpha \kappa \pm \sqrt{\alpha^2 \kappa^2 + 4\kappa \lambda_i}}{2\kappa}$$

由于 κ 和 λ_i 均为正数, 可知这两个根为实根. 式中 M_i, N_i 为待定常数, 可根据板上下表面的电边界条件决定. 此外在式 (40) 中还有

$$S_i = \frac{1}{\lambda_i} \left(s_i + \frac{\kappa \alpha t_i}{\lambda_i} \right), \quad T_i = \frac{t_i}{\lambda_i} \quad (41)$$

于是得电势的解为

$$\begin{aligned} \phi &= \sum_{i=0}^{\infty} Z_i(\zeta) J_0(\sqrt{\lambda_i} \rho) = \\ &\sum_{i=0}^{\infty} (M_i e^{\mu_i \zeta} + N_i e^{\nu_i \zeta} + S_i + T_i \zeta) J_0(\sqrt{\lambda_i} \rho) \end{aligned} \quad (42)$$

5 方程 (42) 的数值方法求解

将 $Z_i(\zeta)$ 在厚度方向离散 (见图 1), 分为 n 层, 在第 j 层中, 令

$$Z_i^{(j)}(\zeta) = Z_i^j N_1^{(j)}(\zeta) + Z_i^{j+1} N_2^{(j)}(\zeta)$$

式中 $N^j(\zeta)$ 和 $N^{j+1}(\zeta)$ 为形函数, 定义为

$$N_1^{(j)}(\zeta) = \frac{\zeta^{j+1} - \zeta}{\Delta \zeta^{(j)}}, \quad N_2^{(j)}(\zeta) = \frac{\zeta - \zeta^j}{\Delta \zeta^{(j)}}$$

其中带括弧的上标 (j) 表示所指示的量在第 j 层内的值, 不带括弧的上标 j 表示所指示的量在第 j 层

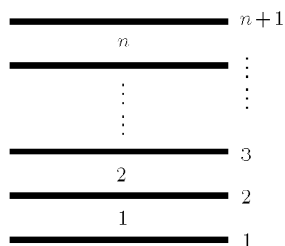


图 1 $Z_i(\zeta)$ 在厚度方向离散

Fig.1 $Z_i(\zeta)$ is discretized across the thickness

界面上的值, $\Delta \zeta^{(j)}$ 为第 j 层的无量纲厚度, $\Delta \zeta^{(j)} = \zeta^{j+1} - \zeta^j$. 于是有

$$\begin{aligned} \int_{-\zeta_1}^{\zeta_2} Z_i' f(\zeta) d\zeta &= [Z_i f(\zeta)]_{-\zeta_1}^{\zeta_2} - \int_{-\zeta_1}^{\zeta_2} f'(\zeta) Z_i d\zeta = \\ &Z_i^{n+1} f(\zeta^{n+1}) - Z_i^1 f(\zeta^1) - \\ &\sum_{j=1}^n \left[\int_{\zeta^j}^{\zeta^{j+1}} N_1^{(j)}(\zeta) f'(\zeta) d\zeta Z_i^j + \right. \\ &\left. \int_{\zeta^j}^{\zeta^{j+1}} N_2^{(j)} f'(\zeta) d\zeta Z_i^{j+1} \right] \end{aligned}$$

记

$$\begin{aligned} c_1^{(j)} &= \int_{\zeta^j}^{\zeta^{j+1}} N_1^{(j)}(\zeta) f'(\zeta) d\zeta \\ c_2^{(j)} &= \int_{\zeta^j}^{\zeta^{j+1}} N_2^{(j)}(\zeta) f'(\zeta) d\zeta \end{aligned}$$

就有

$$\begin{aligned} \int_{-\zeta_1}^{\zeta_2} Z_i' f(\zeta) d\zeta &= [-f(\zeta^1) - c_1^{(1)} - c_2^{(1)} - c_1^{(2)} - \\ &- c_2^{(2)} - c_1^{(3)} - \dots - c_2^{(n-1)} - c_1^{(n)} - \\ &- c_2^{(n)} + f(\zeta^{n+1})][Z_i^1, Z_i^2, \dots, Z_i^n, Z_i^{n+1}]^T = \\ &[C_i]\{Z_i\} \end{aligned}$$

同样有

$$\begin{aligned} \int_{-\zeta_1}^{\zeta_2} Z_i' \zeta f(\zeta) d\zeta &= [Z_i \zeta f(\zeta)]_{-\zeta_1}^{\zeta_2} - \\ &\int_{-\zeta_1}^{\zeta_2} Z_i [f(\zeta) + \zeta f'(\zeta)] d\zeta = \\ &Z_i^{n+1} \zeta^{n+1} f(\zeta^{n+1}) - Z_i^1 \zeta^1 f(\zeta^1) - \\ &\sum_{j=1}^n \left\{ \int_{\zeta^j}^{\zeta^{j+1}} (N_1^{(j)}(\zeta) Z_i^j + N_2^{(j)}(\zeta) Z_i^{j+1}) \cdot \right. \\ &\left. [f(\zeta) + \zeta f'(\zeta)] d\zeta \right\} \end{aligned}$$

记

$$\begin{aligned} d_1^{(j)} &= \int_{\zeta^j}^{\zeta^{j+1}} N_1^{(j)}(\zeta) [f(\zeta) + \zeta f'(\zeta)] d\zeta \\ d_2^{(j)} &= \int_{\zeta^j}^{\zeta^{j+1}} N_2^{(j)}(\zeta) [f(\zeta) + \zeta f'(\zeta)] d\zeta \end{aligned}$$

就有

$$\begin{aligned} \int_{-\zeta_1}^{\zeta_2} Z_i' \zeta f(\zeta) d\zeta &= [-\zeta^1 f(\zeta^1) - d_1^{(1)}, \\ &- d_2^{(1)} - d_1^{(2)}, -d_2^{(2)} - d_1^{(3)}, \dots, -d_2^{(n-1)} - d_1^{(n)}, \end{aligned}$$

$$-d_2^{(n)} + \zeta^{n+1} f(\zeta^{n+1})][Z_i^1, Z_i^2, \dots, Z_i^n, Z_i^{n+1}]^T = [D_i]\{Z_i\}$$

代入式 (38) 得

$$s_i = [L_i]\{Z_i\} + \bar{Y}_i, \quad t_i = \alpha s_i \quad (43)$$

其中

$$[L_i] = \frac{(e_{31}^0)^2 a^2}{C_{11}^0 \lambda_{11}^0 A} \left[\left(\frac{B}{G} - h_2 \right) [C_i] + h [D_i] \right] \quad (44)$$

再令

$$\xi_i = \frac{1}{\lambda_i} \left(1 + \frac{\kappa \alpha^2}{\lambda_i} \right), \quad \eta_i = \frac{\alpha}{\lambda_i} \quad (45)$$

代入式 (41) 得

$$S_i = \xi_i s_i, \quad T_i = \eta_i s_i \quad (46)$$

利用式 (43)~(46), 方程 (40) 可写为

$$Z_i(\zeta) = (\xi_i + \eta_i \zeta) \{ [L_i]\{Z_i\} + \bar{Y}_i \} + M_i e^{\mu_i \zeta} + N_i e^{\nu_i \zeta} \quad (47)$$

取 $\zeta = \zeta^1, \zeta^2, \dots, \zeta^{n+1}$, 相应地式 (47) 有 $n+1$ 个方程

$$Z_i^1 = (\xi_i + \eta_i \zeta^1) ([L_i]\{Z_i\} + \bar{Y}_i) + M_i e^{\mu_i \zeta^1} + N_i e^{\nu_i \zeta^1}$$

$$Z_i^2 = (\xi_i + \eta_i \zeta^2) ([L_i]\{Z_i\} + \bar{Y}_i) + M_i e^{\mu_i \zeta^2} + N_i e^{\nu_i \zeta^2}$$

.....

$$Z_i^{n+1} = (\xi_i + \eta_i \zeta^{n+1}) ([L_i]\{Z_i\} + \bar{Y}_i) +$$

$$M_i e^{\mu_i \zeta^{n+1}} + N_i e^{\nu_i \zeta^{n+1}}$$

或用矩阵和向量形式记为

$$\left([I] - \begin{Bmatrix} \xi_i + \eta_i \zeta^1 \\ \xi_i + \eta_i \zeta^2 \\ \vdots \\ \xi_i + \eta_i \zeta^{n+1} \end{Bmatrix} [L_i] \right) \{Z_i\} = \begin{Bmatrix} \xi_i + \eta_i \zeta^1 \\ \xi_i + \eta_i \zeta^2 \\ \vdots \\ \xi_i + \eta_i \zeta^{n+1} \end{Bmatrix} \bar{Y}_i + M_i \begin{Bmatrix} e^{\mu_i \zeta^1} \\ e^{\mu_i \zeta^2} \\ \vdots \\ e^{\mu_i \zeta^{n+1}} \end{Bmatrix} + N_i \begin{Bmatrix} e^{\nu_i \zeta^1} \\ e^{\nu_i \zeta^2} \\ \vdots \\ e^{\nu_i \zeta^{n+1}} \end{Bmatrix} \quad (48)$$

式中 I 为 $(n+1) \times (n+1)$ 单位阵. 此外上下表面的边界条件还可提供两个方程, 于是 $\{Z_i\}$, M_i 和 N_i 便可解出.

6 算 例

设有功能梯度压电材料圆板, 半径为 20 mm, 厚 2 mm, 板上作用有均布载荷 q , 周边固支, 接地, 板边界电势为 0, 板上、下表面电势为

$$\phi|_{\zeta=\zeta_2} = -\Phi J_0(\sqrt{\lambda_1} \rho), \quad \phi|_{\zeta=-\zeta_1} = \Phi J_0(\sqrt{\lambda_1} \rho) \quad (49)$$

将参考坐标 $z = 0$ 设在板的下表面, 可得中性层坐标为

$$\bar{z} = \frac{\int_0^h z f(z) dz}{\int_0^h f(z) dz} = \frac{\int_0^h z e^{\alpha z/h} dz}{\int_0^h e^{\alpha z/h} dz} = \frac{h(\alpha e^\alpha - e^\alpha + 1)}{\alpha(e^\alpha - 1)}$$

然后再将 $z = 0$ 坐标移至中性轴上. 将关系 $f(z) = e^{\alpha z/h}$ 代入式 (13), (16), (20) 得

$$F(z) = \frac{h}{\alpha} (e^{\alpha z/h} - 1)$$

$$F_1(z) = \frac{h}{\alpha} \left[z e^{\alpha z/h} - \frac{h}{\alpha} (e^{\alpha z/h} - 1) \right]$$

$$G = \frac{h}{\alpha} (e^{\alpha h_2/h} - e^{-\alpha h_1/h})$$

$$A = \left(-\frac{h_1 h^2}{\alpha} - \frac{h^3}{\alpha^2} - \frac{h_1 h^2}{\alpha^2} - \frac{2h^3}{\alpha^3} \right) e^{-\alpha h_1/h} + \left(-\frac{h_2 h^2}{\alpha^2} + \frac{2h^3}{\alpha^3} \right) e^{\alpha h_2/h}$$

$$B = \frac{h^2}{\alpha^2} e^{\alpha h_2/h} - \frac{h^2}{\alpha^2} e^{-\alpha h_1/h} (1 + \alpha)$$

将板上载荷 $q(r) = q$ 代入式 (24) 和 (35) 得

$$Q_1 = \frac{qa^2}{8} \quad (50)$$

$$\frac{\bar{e}_{31}^0}{C_{11}^0 A} \left[\frac{qa^2 \rho^2}{4} - D_2 \right] = \sum_{i=0}^{\infty} Y_i J_0(\sqrt{\lambda_i} \rho) \quad (51)$$

其中 D_2 见式 (23). 由此得到 Bessel 级数系数的表达式

$$Y_i = -\frac{2\bar{e}_{31}^0}{J_1^2(\sqrt{\lambda_i}) C_{11}^0 A} \int_0^1 \left(\frac{qa^2 \rho^3}{4} - \rho D_2 \right) J_0(\sqrt{\lambda_i} \rho) d\rho$$

利用关系

$$\int_0^1 \rho^3 J_0(\sqrt{\lambda_i} \rho) d\rho = \frac{J_1(\sqrt{\lambda_i})}{\sqrt{\lambda_i}} - \frac{2J_2(\sqrt{\lambda_i})}{\lambda_i}$$
$$\int_0^1 \rho J_0(\sqrt{\lambda_i} \rho) d\rho = \frac{J_1(\sqrt{\lambda_i})}{\sqrt{\lambda_i}}$$

和式 (39), 可得无量纲 Bessel 系数的表达式

$$\bar{Y}_i = \frac{a^2}{\bar{\lambda}_{11}^0 \Phi} Y_i = - \frac{2a^2 \bar{e}_{31}^0}{J_1^2(\sqrt{\lambda_i}) \bar{C}_{11}^0 \bar{\lambda}_{11}^0 \Phi A} \cdot \left[\frac{qa^2}{4} \left(\frac{J_1(\sqrt{\lambda_i})}{\sqrt{\lambda_i}} - \frac{2J_2(\sqrt{\lambda_i})}{\lambda_i} \right) - D_2 \frac{J_1(\sqrt{\lambda_i})}{\sqrt{\lambda_i}} \right] \quad (52)$$

在板厚方向分层以后, 可用式 (48) 求解层面上的 $\{Z_i\}$ 和系数 M_i, N_i , 再用式 (40) 求得 $Z_i(\zeta)$, 便可由式 (32) 求得电势 ϕ , 再由式 (22) 求转角 β . 注意到式 (48) 和式 (22) 中都含有未知的 D_2 , 需要用迭代法求解. 可以先假设 D_2 , 然后通过上述过程求得 ϕ 后, 再用式 (23)~(25) 求解新的 D_2 , 这样反复迭代直至假设的 D_2 与解出的 D_2 充分接近为止. 实际上迭代过程收敛很快, 一般迭代 3 次就可使误差小于 0.0001.

在中性层 $z = 0$ 处的材料常数如表 1 所示.

表 1 材料常数

Table 1 Material constants

C_{ij}/GPa		$e_{ij}/(\text{C}\cdot\text{m}^{-2})$	$\lambda_{ij}/(\text{F}\cdot\text{m}^{-1})$
$C_{11}^0 = C_{22}^0 = 74.1$	$C_{33}^0 = 83.6$	$e_{31}^0 = e_{32}^0 = -0.16$	$\lambda_{11}^0 = \lambda_{22}^0 = 8.25 \times 10^{-11}$
$C_{12}^0 = 45.2$	$C_{13}^0 = C_{23}^0 = 39.3$	$e_{33}^0 = 0.347$	$\lambda_{33}^0 = 9.02 \times 10^{-11}$
$C_{44}^0 = C_{55}^0 = 13.17$	$C_{66}^0 = 14.45$	$e_{24}^0 = 0$	

把板沿厚度方向均分为 10 层, 得每层厚度 $\Delta\zeta^{(j)} = 0.1, (j = 1, 2, \cdots, 10)$. 取 $q = 1.0\text{ kPa}$, $\Phi = 1.0\text{ kV}$, 图 2 (a)~(e) 分别给出了在 5 种材料梯度情况下, $\rho = 0, 0.2, 0.4, 0.6, 0.8$ 处电势沿板厚方向的变化情况. 为了描述方便, 仍然把坐标 $\zeta = 0$ 放置在板的下表面. 可以看出, 当 $\alpha < 0$ 时, 靠近上表面的电势要比靠近下表面的沿板厚方向变化要快, 当 $\alpha > 0$ 时情况则刚好相反, 当 $\alpha = 0$ 时电势沿板厚度线性分布. 这与文献 [1] 的结果趋势一致.

图 3 给出了转角 β 在 5 种材料梯度情况下沿板半径方向的分布, β 在板中心和边界上都为 0, 符合板周边固支的边界条件. 可以看到在这里给定的载

荷下, 转角还是相当小的. 中性层上的剪应力可由式 (15) 计算, 它的分布可见图 4. 可以看出中性层上的剪应力并非随 α 的增加而单调增加, 在 $\alpha > -0.5$ 以后随 α 的增加而减小.

在求解过程中板挠度 w 始终未出现, 而我们又假设了板中性层的法线在板变形后不一定垂直于中性层, 因此由 β 计算 w 还需要附加条件. 这里我们可以合理地认为, β 与 $-\frac{dw}{dr}$ 之差就是中性层上的剪应变, 亦即

$$\frac{dw}{dr} = -\beta + \frac{\tau_{rz}|_{z=0}}{C_{44}^0}$$

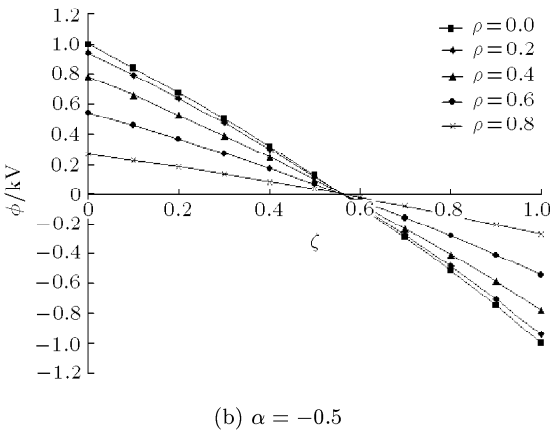
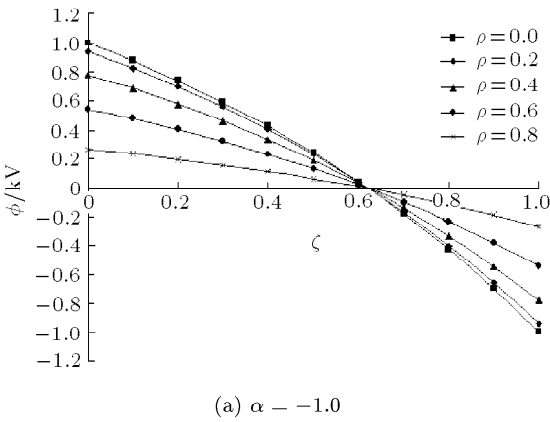
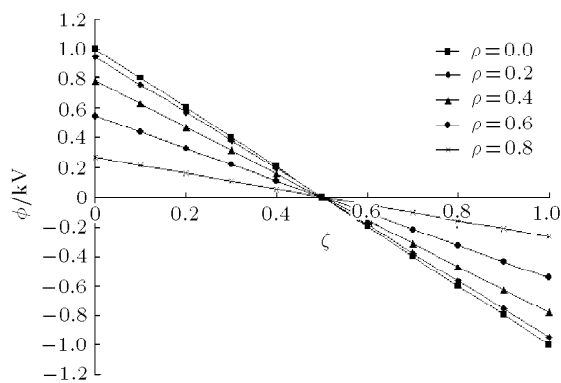
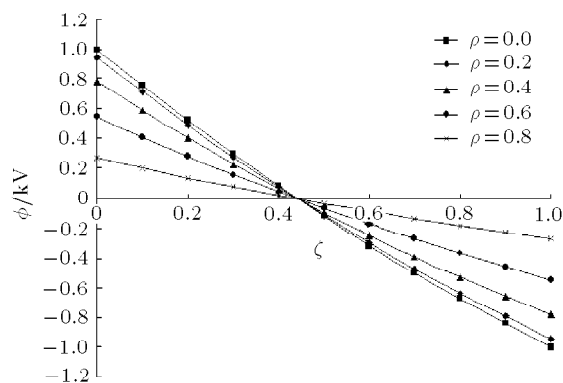
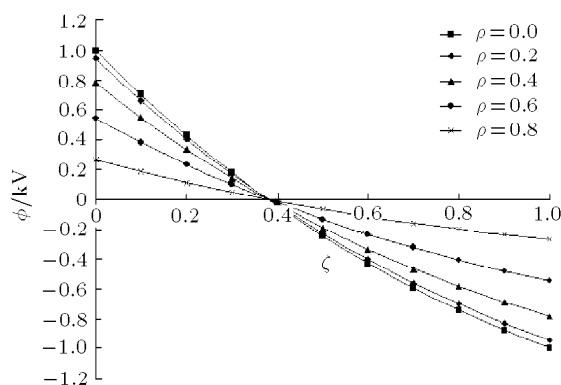
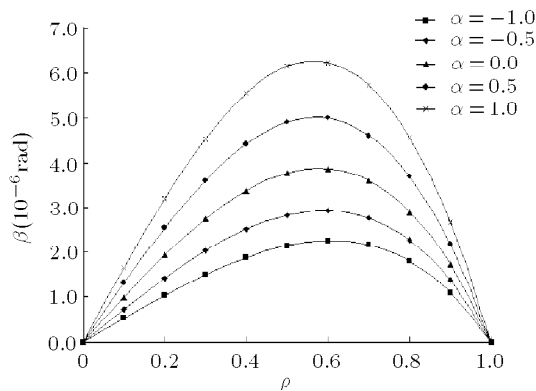
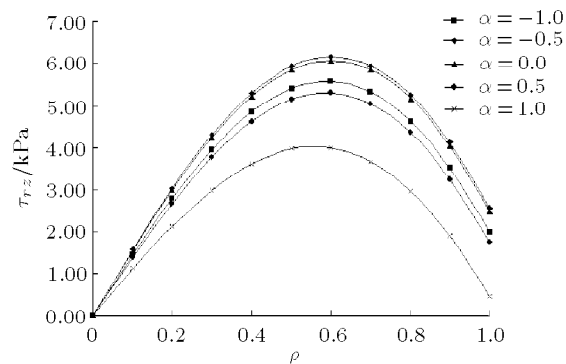


图 2 电势 ϕ 沿板厚方向的分布

Fig.2 Distribution of electric potential ϕ across the thickness

(c) $\alpha = 0.0$ (d) $\alpha = 0.5$ (e) $\alpha = 1.0$ 图 2 电势 ϕ 沿板厚方向的分布 (续)Fig.2 Distribution of electric potential ϕ across the thickness (continued)图 3 转角 β 沿半径方向的分布Fig.3 Distribution of slope β along the radius图 4 中性层上的剪应力 τ_{rz} 沿半径方向的分布Fig.4 Distribution of the shear stress τ_{rz} in the neutral layer along the radius

于是可以从边界上 $w = 0$ 开始, 一步步算得沿半径方向的挠度分布, 如图 5 所示. 可以看到挠度也是很小的.

7 方法的验证讨论

为了验证方法的正确性, 设 $e_{31}^0 = e_{33}^0 = e_{15}^0 = 0$,

即不考虑力电耦合效应的影响, 由式 (22)~(29) 知, 在板周边固支时

$$\beta = -\frac{1}{\bar{C}_{11}^0 A r} \left[\int_0^r r \left(\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right) dr - \frac{r^2}{a^2} \int_0^a r \left(\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right) dr \right] \quad (53)$$

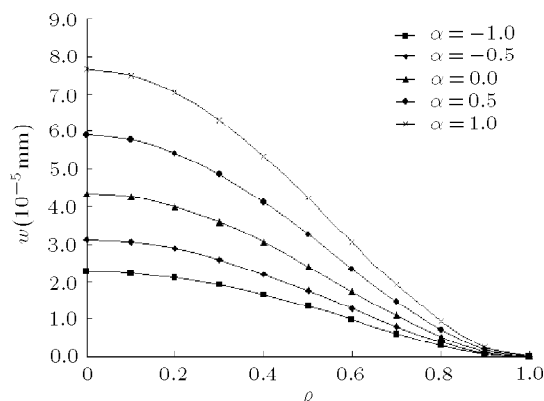


图 5 挠度沿半径方向的分布

Fig.5 Distribution of deflection of the plate along the radius

在板周边简支时

$$\beta = -\frac{1}{\bar{C}_{11}^0 A r} \left\{ \int_0^r r \left[\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right] dr - \frac{\bar{C}_{11}^0 r^2}{\bar{C}_{11}^0 + \bar{C}_{12}^0} \int_0^a \frac{1}{r} \left[\int_0^r r q(r) dr \right] dr - \frac{\bar{C}_{12}^0 - \bar{C}_{11}^0 r^2}{\bar{C}_{11}^0 + \bar{C}_{12}^0 a^2} \int_0^a r \left[\int_0^r \frac{1}{r} \left(\int_0^r r q(r) dr \right) dr \right] dr \right\} \quad (54)$$

这是梯度弹性薄板弯曲问题的解析解. 进一步设 $\alpha = 0$, 即材料是均质的; 再考虑材料是各向同性的. 容易证明, 式 (53) 和 (54) 完全退化为经典的弹性薄圆板轴对称问题的解析解.

为了验证电势解的正确性, 用轴对称有限单元法进行了计算. 所取的模型尺寸、载荷以及材料常数和算例中完全一样. 在子午面上沿半径方向分 20 份, 厚度方向分 10 份, 形成 200 个单元的矩形网格. 采用 4 节点的轴对称等参单元. 用 FORTRAN 语言编程计算, 所得电势的解与图 4 表示的结果几乎完全相同. 如果绘成曲线, 则看不出两者的差别.

8 结 论

本文引入了 Reissner-Mindlin 假设, 即板中性面的法线在板变形后仍为直线, 但不一定与中性面垂直, 忽略了 3 个次要应力 σ_z , τ_{rz} , $\tau_{\theta z}$ 对变形和电位移的影响, 提出了功能梯度圆板的简化理论. 又假设材料常数在板厚方向按指数规律变化, 推导了板的周边固定或简支同时又接地情况下中性层法线转角的解和用 Fourier-Bessel 级数表示的电势解. 这个解在形式上比精确解简单得多, 进行数值计算时也相当方便与快捷. 在算例中给出了板的周边固定、

接地情况下的计算结果并进行了分析讨论. 最后验证了解的正确性, 也说明了本文所做的一些简化假设是合理的.

本文虽然研究的是功能梯度圆板的轴对称问题, 但这里的简化假设和推导方法可以推广到其他的类似问题上去, 如承受任意载荷的圆板, 矩形板等. 当然解析解的形式会更复杂些.

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SIMPLIFIED THEORY AND ANALYTICAL SOLUTIONS FOR FUNCTIONALLY GRADED PIEZOELECTRIC CIRCULAR PLATE¹⁾

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Abstract Simplified theory and analytical solutions for a functionally graded piezoelectric circular plate subjected to axisymmetric loading was presented with some assumptions, such as Kirchhoff assumption, Reissner-Mindlin assumption, and the assumption that material properties vary exponentially across the thickness of the plate. We derived the solution of the rotation angle of the normal line of the neutral layer and the solution of the electric potential in the plate expressed by Fourier-Bessel series, in the cases that the edges of the plate was fixed or simply supported and grounded. The presented solutions were obviously simple compared with some exact analytical solutions and easy to perform the numerical analysis. The numerical results for the edge-fixed/grounded plate were applied to validate the simplified theory presented in the paper.

Key words functionally graded material, axisymmetric, circular plate, simplified theory, analytical solution

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