带源参数的二维热传导反问题的无网格方法 1)

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摘要 利用无网格有限点法求解带源参数的二维热传导反问题，推导了相应的离散方程．与其它基于网格的方法相比，有限点法采用移动最小二乘法构造形函数，只需要节点信息，不需要划分网格，用配点法离散控制方程，可以直接施加边界条件，不需要在区域内部求积分．用有限点法求解二维热传导反问题具有数值实现简单、计算量小、可以任意布置节点等优点．最后通过算例验证了该方法的有效性．

关键词 无网格方法，有限点法，移动最小二乘法，热传导反问题，源参数

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引 言

热传导反问题出现在各种各样的工程领域，引起众多学者的广泛关注．

目前对热传导反问题的大量研究都集中于解的唯一性和存在性等方面，而且对反问题的数值解一般都是采用基于网格的数值方法，比如有限差分法、有限元法等 [1,2]．与以往基于网格的方法不同，无网格方法采用基于点的近似，不需要在求解区域内划分用来确定插值函数的网格，为科学和工程计算带来很大的方便 [3~9]．有限点法 (the finite point method, 即 FPM)[9] 是采用移动最小二乘法 [3] 形成近似函数、用配点法离散控制方程组的一种无网格方法．

本文考虑了用有限点法求解一类带源参数的二维半线性热传导反问题．和其它基于网格的方法相比，用有限点法求解热传导反问题具有明显的优势，它用移动最小二乘法构造近似函数，只需要节点信息，不需要网格划分．用配点法离散求解方程组，可以直接施加边界，不需要在区域内部求积分，减少了计算量．最后通过算例验证了该方法求二维热传导反问题的有效性．

1 二维热传导反问题

考虑如下的求源参数 $p(t)$ 的反问题

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+p(t)u+\varphi(x,y,t)\\ &0\le x,\quad y\le 1,\quad 0<t\le T\end{aligned}\right\}\tag{1}$$

初始条件为

$$u(x,y,0)=f(x,y)\quad (0\le x\,,\ y\le 1)\tag{2}$$

边界条件为

$$u(0,y,t)=g_0(y,t)\quad (0<t\le T\,,\ 0\le y\le 1)\tag{3}$$

$$u(1,y,t)=g_1(y,t)\quad (0<t\le T\,,\ 0\le y\le 1)\tag{4}$$

$$u(x,0,t)=h_0(x,t)\quad (0<t\le T\,,\ 0\le x\le 1)\tag{5}$$

$$u(x,1,t)=h_1(x,t)\quad (0<t\le T\,,\ 0\le x\le 1)\tag{6}$$

并且

$$\left.\begin{aligned} &u(x_0,y_0,t)=E(t)\\ &(x_0,y_0)\in(0,1)\times(0,1)\,,\quad 0<t\le T\end{aligned}\right\}\tag{7}$$

其中 $f, g_0, g_1, h_0, h_1, \varphi, E$ 为已知函数； u, p 是未知函数．

方程 (1)~(7) 描述了带热源 $p(t)$ 的热传导过程．

利用如下变换 [1]

$$r(t)=\exp\Big(-\int_0^t p(s)\mathrm{d}s\Big)\tag{8}$$

$$w(x,y,t)=r(t)u(x,y,t)\tag{9}$$

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方程 (1) 变为

$$\left.\begin{array}{c} w_t = w_{xx} + w_{yy} + r(t)\varphi(x,y,t) \\ 0 \le x\,,\ y \le 1\,,\ 0 < t \le T \end{array}\right\} \tag{10}$$

初始条件

$$w(x,y,0) = f(x,y)\,,\ 0 \le x\,,\ y \le 1 \tag{11}$$

边界条件

$$w(0,y,t) = r(t)g_0(y,t)\ (0 < t \le T\,,\ 0 \le y \le 1) \tag{12}$$

$$w(1,y,t) = r(t)g_1(y,t)\ (0 < t \le T\,,\ 0 \le y \le 1) \tag{13}$$

$$w(x,0,t) = r(t)h_0(x,t)\ (0 < t \le T\,,\ 0 \le x \le 1) \tag{14}$$

$$w(x,1,t) = r(t)h_1(x,t)\ (0 < t \le T\,,\ 0 \le x \le 1) \tag{15}$$

$r(t)$ 可以由下式得到

$$r(t) = \frac{w(x_0,y_0,t)}{E(t)} \tag{16}$$

经过这个变换，源参数消失了，这样我们可以把式 (10)∼(16) 作为正问题来求解.

2 二维热传导反问题的有限点法

在问题所在域 $[0,1]\times[0,1]$ 上均匀布置 n 个节点 (x_i,y_i), $i=1,2,\cdots,n$. 于是 $w(x,y,t)$ 的逼近函数表示为

$$w^h(x,y,t) = \sum_{i=1}^{n} \varPhi_i(x,y)\lambda_i(t) \tag{17}$$

其中 $\varPhi_i(x,y)$ 为移动最小二乘形函数 [3]. 将近似函数式 (17) 分别代入式 (10) 和式 (11) 得到

$$\begin{aligned} &\sum_{i=1}^{n} \varPhi_i(x,y)\frac{\mathrm{d}\lambda_i(t)}{\mathrm{d}t} = \\ &\quad \sum_{i=1}^{n}\Big(\frac{\partial^2 \varPhi_i(x,y)}{\partial x^2} + \frac{\partial^2 \varPhi_i(x,y)}{\partial y^2}\Big)\lambda_i(t) + \\ &\quad \sum_{i=1}^{n}\frac{\varPhi_i(x_0,y_0)}{E(t)}\varphi(x,y,t)\lambda_i(t) \end{aligned} \tag{18}$$

$$\sum_{i=1}^{n} \varPhi_i(x,y)\lambda_i(0) = f(x,y) \tag{19}$$

式 (18) 和 (19) 对每一个节点 (x_i,y_i), $i=1,2,\cdots,n$ 成立，从而可以写成矩阵形式为

$$\boldsymbol{H}\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = (\boldsymbol{H}_1 + \boldsymbol{C})\boldsymbol{M} \tag{20}$$

$$\boldsymbol{M}(0) = \boldsymbol{H}^{-1}\boldsymbol{G} \tag{21}$$

其中

$$\boldsymbol{M} = (\lambda_1(t),\lambda_2(t),\cdots,\lambda_n(t))^{\mathrm{T}} \tag{22}$$

$$\boldsymbol{G} = (f(x_1,y_1), f(x_2,y_2),\cdots,f(x_n,y_n))^{\mathrm{T}} \tag{23}$$

$$\boldsymbol{H} = \begin{bmatrix} \varPhi_1(x_1,y_1) & \varPhi_2(x_1,y_1) & \cdots & \varPhi_n(x_1,y_1) \\ \varPhi_1(x_2,y_2) & \varPhi_2(x_2,y_2) & \cdots & \varPhi_n(x_2,y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varPhi_1(x_n,y_n) & \varPhi_2(x_n,y_n) & \cdots & \varPhi_n(x_n,y_n) \end{bmatrix} \tag{24}$$

$$\boldsymbol{H}_1 = \begin{bmatrix} \varPhi_{1,xx}(x_1,y_1) & \varPhi_{2,xx}(x_1,y_1) & \cdots & \varPhi_{n,xx}(x_1,y_1) \\ \varPhi_{1,xx}(x_2,y_2) & \varPhi_{2,xx}(x_2,y_2) & \cdots & \varPhi_{n,xx}(x_2,y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varPhi_{1,xx}(x_n,y_n) & \varPhi_{2,xx}(x_n,y_n) & \cdots & \varPhi_{n,xx}(x_n,y_n) \end{bmatrix} + \begin{bmatrix} \varPhi_{1,yy}(x_1,y_1) & \varPhi_{2,yy}(x_1,y_1) & \cdots & \varPhi_{n,yy}(x_1,y_1) \\ \varPhi_{1,yy}(x_2,y_2) & \varPhi_{2,yy}(x_2,y_2) & \cdots & \varPhi_{n,yy}(x_2,y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varPhi_{1,yy}(x_n,y_n) & \varPhi_{2,yy}(x_n,y_n) & \cdots & \varPhi_{n,yy}(x_n,y_n) \end{bmatrix} \tag{25}$$

$$\boldsymbol{C} = \frac{1}{E(t)}\begin{bmatrix} \varPhi_1(x_0,y_0)\varphi(x_1,y_1,t) & \varPhi_2(x_0,y_0)\varphi(x_1,y_1,t) & \cdots & \varPhi_n(x_0,y_0)\varphi(x_1,y_1,t) \\ \varPhi_1(x_0,y_0)\varphi(x_2,y_2,t) & \varPhi_2(x_0,y_0)\varphi(x_2,y_2,t) & \cdots & \varPhi_n(x_0,y_0)\varphi(x_2,y_2,t) \\ \vdots & \vdots & \ddots & \vdots \\ \varPhi_1(x_0,y_0)\varphi(x_n,y_n,t) & \varPhi_2(x_0,y_0)\varphi(x_n,y_n,t) & \cdots & \varPhi_n(x_0,y_0)\varphi(x_n,y_n,t) \end{bmatrix} \tag{26}$$

将式 (20) 和 (21) 看成一个常微分方程组，对式 (20) 进行时间离散可得

$$\boldsymbol{H}\frac{\boldsymbol{M}^{(k+1)}-\boldsymbol{M}^{(k)}}{\Delta t}=(\boldsymbol{H}_1+\boldsymbol{C})\frac{\boldsymbol{M}^{(k+1)}+\boldsymbol{M}^{(k)}}{2} \tag{27}$$

即

$$(2\boldsymbol{H}-\Delta t(\boldsymbol{H}_1+\boldsymbol{C}))\boldsymbol{M}^{(k+1)}=(2\boldsymbol{H}+\Delta t(\boldsymbol{H}_1+\boldsymbol{C}))\boldsymbol{M}^{(k)} \tag{28}$$

利用初始条件 $\boldsymbol{M}(0)=\boldsymbol{M}^{(0)}=\boldsymbol{H}^{-1}\boldsymbol{G}$ 以及边界条件 (12)~(15) 可以求得 $\boldsymbol{M}^{(1)},\boldsymbol{M}^{(2)},\cdots,\boldsymbol{M}^{(k)}$, 即

$$\boldsymbol{M}^{(k)}=\boldsymbol{M}(t_k)=(\lambda_1(t_k),\lambda_2(t_k),\cdots,\lambda_n(t_k)) \tag{29}$$

根据式 (16), $r(t)$ 的近似函数可以按如下求得

$$r^h(t)=\frac{w^h(x_0,y_0,t)}{E(t)} \tag{30}$$

从而得到方程 (1)~(7) 的近似解

$$u^h(x,y,t)=\frac{w^h(x,y,t)}{r^h(t)} \tag{31}$$

最后根据式 (8) 得到源参数 $p(t)$ 的近似解为

$$p^h(t)=-\frac{\mathrm{d}r^h(t)/\mathrm{d}t}{r^h(t)} \tag{32}$$

3 数值算例

为了验证本文提出的求解二维半线性热传导反问题的有限点法的有效性，本节给出了具体的算例.

考虑问题 (1)~(7)[10], 其中

$$f(x,y)=\sin(\pi x)\cos(\pi y) \tag{33}$$

$$g_0(0,y,t)=0 \tag{34}$$

$$g_1(1,y,t)=0 \tag{35}$$

$$h_0(x,0,t)=\exp(-t)\sin(\pi x) \tag{36}$$

$$h_1(x,\,1,\,t)=-\exp(-t)\sin(\pi x) \tag{37}$$

$$\varphi(x,y,t)=(2\pi^2-t^2-2)\exp(-t)\sin(\pi x)\cos(\pi y) \tag{38}$$

$$E(t)=\frac{1}{2}\exp(-t) \tag{39}$$

$$(x_0,y_0)=(0.25,\,0.25) \tag{40}$$

精确解为

$$u(x,y,t)=\exp(-t)\sin(\pi x)\cos(\pi y) \tag{41}$$

$$p(t)=1+t^2 \tag{42}$$

我们利用有限点法对上述反问题进行求解. 如图 1 所示，在求解域内均匀布置了 121 个节点. 权函数选为 Gauss 权函数，基函数取线性基，$\Delta t=0.000\,1$. 图 2 给出了源参数 $p(t)=t^2+1$ 的解析解和数值解. 图 3 给出了函数 $u(x,t)=\exp(-t^2)(\cos(\pi x)+\sin(\pi x))$ 在 $t=0.1$, 0.3, 0.5, 0.7 和 1 时刻 $y=0.6$ 处的解析解和数值解. 图 4 给出了函数 $u(x,t)=\exp(-t^2)(\cos(\pi x)+\sin(\pi x))$ 在 $t=0.1$, 0.3, 0.5, 0.7 和 1 时刻 $x=0.5$ 处的解析解和数值解. 通过图 2~图 4 可以看出，用有限点法求热传导反问题得到的数值解和解析解吻合得很好.

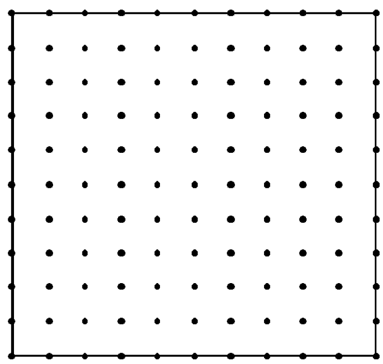

图 1 在 $[0,1]\times[0,1]$ 内的节点分布

Fig.1 The nodes distribution on $[0,1]\times[0,1]$

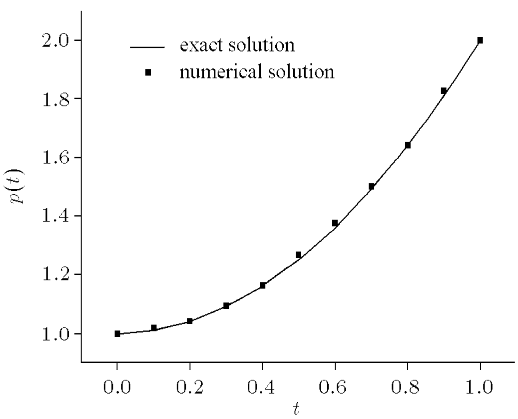

图 2 源参数 $p(t)$ 的数值解和解析解

Fig.2 The numerical and exact solutions of $p(t)$

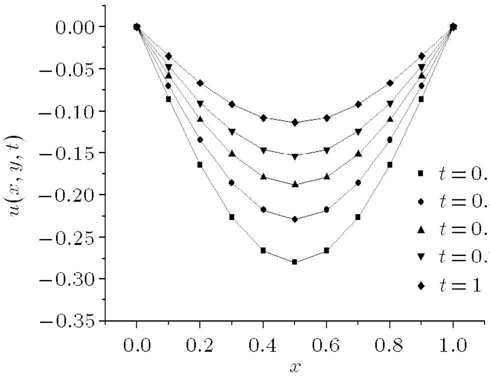

图 3 $u(x,y,t)$ 的数值解和解析解

Fig.3 The numerical and exact solutions of $u(x,y,t)$

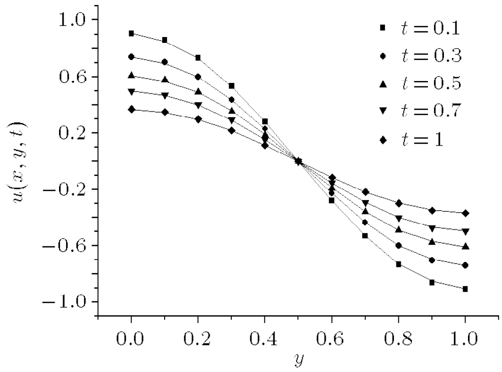

图 4 $u(x,y,t)$ 的数值解和解析解

Fig.4 The numerical and exact solutions of $u(x,y,t)$

4 结 论

本文利用有限点法求解带源参数的二维热传导反问题．和其它基于网格的方法，如有限元和边界元等数值方法相比，有限点法在构造近似函数的时候，不需要划分网格，只需要节点，对需要进行网格重构和具有复杂区域的问题而言，有限点法具有较大的优势．用配点法来离散方程组，可以直接施加边界条件，不需要在区域内求积分，大大减少了计算量．用有限点法求解带源参数的二维热传导反问题具有可以任意布置节点、实现简单、计算量小等优点．

参 考 文 献

1 Cannon JR, Lin Y, Xu S. Numerical procedure for the determination of an unknown coefficient in semilinear parabolic partial differential equations. *Inverse Problems*, 1994, 10: 227～243

2 Macbain JA, Bendar JB. Existence and uniqueness properties for one-dimensional magnetotelluric inverse problem. *J Math Phys*, 1986, 27: 645～649

3 Belytschko T, Krongauz Y. Meshless methods: an overview and recent developments. *Computer Methods in Applied Mechanics and Engineering*, 1996, 139: 3～47

4 程玉民,彭妙娟,李九红.复变量移动最小二乘法及其应用.力学学报，2005, 37(6): 719～723 (Cheng Yumin, Peng Miaojuan, Li Jiuhong. The complex variable moving least-square approximation and its application. *Chinese Journal of Theoretical and Applied Mechanics*, 2005, 37(6): 719～723 (in Chinese))

5 程玉民,李九红.弹性力学的复变量无网格方法.物理学报, 2005, 54(10): 4463～4471 (Cheng Yumin, Li Jiuhong. A meshless method with complex variables for elasticity. *Acta Physica Sinica*, 2005, 54(10): 4463～4471 (in Chinese))

6 程玉民，陈美娟．弹性力学的一种边界无单元法．力学学报，2003, 35(2): 181～186 (Cheng Yumin, Chen Meijuan. A boundary element-free method for linear elasticity. *Acta Mechanica Sinica*, 2003, 35(2): 181～186 (in Chinese))

7 程玉民，彭妙娟. 弹性动力学的边界无单元法. 中国科学，2005, 35(4): 435～448 (Cheng Yumin, Peng Miaojuan. Boundary element-free method for elastodynamics. *Science in China*, 2005, 35(4): 435～448 (in Chinese))

8 秦义校，程玉民．弹性力学的重构核粒子边界无单元法．物理学报，2006, 55(7): 3215～3222 (Qin Yixiao, Cheng Yumin. Reproducing kernel particle boundary element-free method for elasticity. *Acta Physica Sinica*, 2006, 55(7): 3215～3222(in Chinese))

9 Onate E, Idelsohn SR, Zienkiewicz OC. A finite point method in computational mechanics. Applications to convective transport and fluid flow. *International Journal for Numerical Methods in Engineering*, 1996, 39: 3839～3866

10 Barran EC, Fatullayev AG. Determination of an unknown source parameter in two-dimensional heat equation. *Applied Mathematics and Computation*, 2004, 159: 881～886

THE MESHLESS METHOD FOR A TWO-DIMENSIONAL INVERSE HEAT CONDUCTION PROBLEM WITH A SOURCE PARAMETER [1)]

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Abstract Inverse problems are widely found in the aerospace, nuclear physics, metallurgy and other fields. The finite difference method and the finite element method are main numerical methods to obtain numerical solutions for inverse problems. The finite point method is a meshless method. Comparing with the numerical methods based on mesh, such as finite element method and boundary element method, the finite point method only uses scattered nodes without having to mesh the domain of the problem when the shape function is formed. In this paper, the finite point method is used to obtain numerical solutions of two-dimensional inverse heat conduction problems with a source parameter, and the corresponding discretized equations are obtained. The collocation method is used to discretize the governing partial differential equations, and boundary conditions can be directly enforced without numerical integration in the problem domain. This reduces the computation cost greatly. A numerical example is given to show the effectiveness of the method. The finite point method can also be applied to other inverse problems.

Key words meshless method, the finite point method, the moving least-square approximation, inverse heat conduction problem, source parameter

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