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RESEARCH PAPER

The effects of relative density of metal foams on the stresses and deformation of beam under bending

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Abstract The exact analytic solution of the pure bending beam of metallic foams is given. The effects of relative density of the material on stresses and deformation are revealed with the Triantafillou and Gibson constitutive law (TG model) taken as the analysis basis. Several examples for individual foams are discussed, showing the importance of compressibility of the cellular materials. One of the objects of this study is to generalize Hill's solution for incompressible plasticity to the case of compressible plasticity, and a kinematics parameter is brought into the analysis so that the velocity field can be determined.

Keywords Metal foams · Relative density · Compressible plasticity · Constitutive law · TG model

1 Introduction

The low density metallic foam is a new class of engineering materials with promising mechanical, thermal, electrical and acoustical properties. A wide range of applications is currently under exploration, including ultra-light structural component in air and sea vehicles to energy absorbers in automobile and packing industries, heat dissipation media for high power electronics and sound absorbing devices for noise and vibration control.

Foam materials consist of metal, ceramic and polymeric foam. In this study, discussion is focused on the metal foam, which is of ductile-plastic cellular materials. The plastic response of metal foams differs fundamentally from that of fully dense metals, because the foam compacts when compressed, and the yield criterion is dependent on mean stress or hydrostatic pressure. For their extensive application in practice, it is necessary to understand fully the strength of cellular materials in engineering designs in which the mechanical properties, including the constitutive relation, are of substantial importance.

Since the publication of monographs on cellular solid structure and properties given by Gibson and Ashby [1], the study on constitutive laws under multi-axial loading condition, of the material has been widely conducted.

The presence of the sub-structure—cells leads to specific constitutive relation representing rate-dependent compressible plasticity. The mean stress σ_m or the hydrostatic pressure p must be included into the constitutive equation.

Gibson et al. [2] proposed the first yield/loading surface equation for metal foams on the basis of the flow rule and thus the corresponding constitutive law (which is briefly called the GAZT model) can be obtained. Soon afterwards, Triantafillou and Gibson [3] proposed another constitutive law for the material (briefly called the TG model). In these two constitutive models, the relative density ρ^*/ρ_s presents a new material parameter incorporated in the constitutive law with ρ^* denoting the density of cellular material and ρ_s one of the cell wall material. Recently Miller [4] and Deshpande and Fleck [5,6] made a series of experimental and theoretical studies on the constitutive law. Ashby et al. [7] gave a comprehensive description for different aspects of the material in their monograph, and recommended DF

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constitutive law. All of these constitutive models present a common feature that the material is modeled as an equivalent continuous medium with some new material parameters. Of the constitutive law of continuous model and their applications, some review papers can be found in [8], and Fan and coworkers developed some simplified crack models and perturbation methods to construct several analytic solutions for crack problems of the material [9].

Apart from the crack problem, the conventional structures problems, especially the St Venant problems for the material are also interesting for engineering applications. Here we give a description of pure bending beams made of cellular materials based on the TG rigid-hardening constitutive law. This constitutive equation is simpler, and helpful for constructing some analytic solutions. In addition, it is rather convenient for revealing the effects of size and geometry configuration of cells, because the relative density ρ^*/ρ_s is closely connected with these structure factors. In the classical plasticity holding true for fully dense material, there are some pure bending solutions for rigid-perfect plasticity, linear hardening and power-law hardening responses [10–12], thus providing a basis for comparing and checking calculation results for the cellular materials.

In this paper, the TG constitutive model is used to figure out the pure bending behavior of a beam made of rigid-hardening metal foam materials, and the emphasis is laid on introducing a kinematics parameter $\dot{\theta}$ and obtaining exact solutions of the strain rate and velocity field. The present paper is organized as follows. In Sect. 2, the pure bending problem of a beam made of foam materials is described, and the TG constitutive model is introduced in Sect. 3. In Sect. 4, the stresses of the problem are calculated and some pictures about the stress variation verses height of the beam for different relative density ρ^*/ρ_s are given. In Sect. 5, a kinematics parameter is introduced into the problem and exact solutions of the strain rate and velocity field are derived. Conclusions and discussions on the obtained results are drawn in Sects. 6 and 7.

2 Statement of the problem

Consider the bending of a uniform rectangular beam with internal radii a, neutral fiber layer c and external radii b subjected to a couple M at its terminals as sketched in Fig. 1. The beam is assumed to be made from a rigid-hardening foam material with yield strength σ^*_{pl}. From Fig. 1 we can see that the bending beam can be divided into two regions. If the radius r is greater than the radius c of the neutral layer, the material is in tensile, otherwise the material is compressed, and both of them are discussed respectively in the following.

The advantage in the coordinate system shown in Fig. 1 is that the constitutive equation would be simpler and the

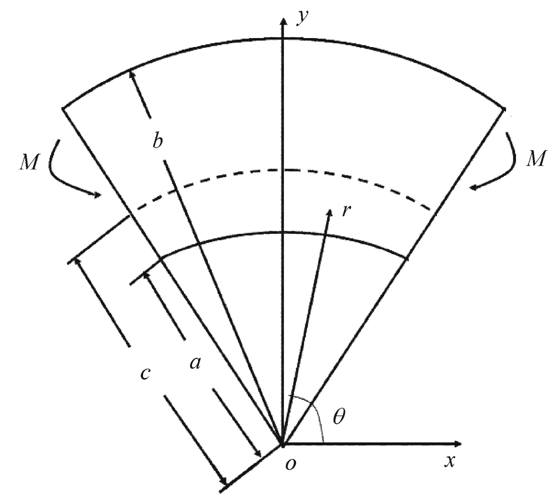

Fig. 1 Polar coordinate system for the beam after bending

analysis of stress field be independent on the velocity field (i.e. the stress field can be statically determined).

3 Constitutive equations

The existence of cells distinguished the cellular materials from conventional dense materials. The effect of cells rests with the so-called compressible plasticity, which leads to the requirement that the mean stress $\sigma_m = \frac{\sigma_{kk}}{3}$ or the hydrostatic pressure $p = -\frac{\sigma_{kk}}{3}$ must be included in the yield criterion and the constitutive equation. Triantafillou and Gibson [3] proposed a simplified model (called the TG model), which could reveal the effects of size and geometry structure of the cells. On the other hand, it is simple and can help us to construct some analytic solutions.

In the following we would like to introduce a constitutive equations based on the TG model. It is well-known that the yield/loading surface can be expressed by

$$\Phi = \hat{\sigma} - Y = 0, \tag{1}$$

where $\hat{\sigma}$ is the TG generalized effective stress defined by

$$\hat{\sigma} = \sigma_e + 0.03\frac{\rho^*}{\rho_s}\sigma_m, \tag{2}$$

where σ_e the von Mises effective stress

$$\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}}. \tag{3}$$

Except for some cases, we can also take σ_e as the Tresca effective stress and s_{ij} the stress deviatoric stress tensor

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \tag{4}$$

where σ_{ij} is the stress tensor, δ_{ij} the Kronecker delta, and the mean stress $\sigma_m = \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz})$. In Eq. (1) if

$$Y = \sigma^*_{pl}, \tag{5a}$$

where σ^*_{pl} is the uniaxial yield strength, then Eq. (1) represents the initial yield surface. If

$$Y = Y(h), \tag{5b}$$

in which h is a parameter for describing the plastic deformation history, then Eq. (1) is the evolution equation of the yield/loading surface.

For self-similar isotropic rigid-hardening behavior, according to the flow rule in plasticity theory we can obtain the constitutive equation based on the yield/loading surface listed above as:

$$\dot{\varepsilon}_{ij} = \frac{\dot{\hat{\sigma}}}{H(\hat{\sigma})}\frac{\partial \Phi}{\partial \sigma_{ij}}, \tag{6}$$

where $\dot{\varepsilon}_{ij}$ denotes the strain rate tensor, $H(\hat{\sigma})$ denotes the hardening modulus and can be obtained by a simple uniaxial tensor experiment, i.e. $H(\hat{\sigma}) = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon^p}$. In addition, from the initial yield surface equation we can easily reduce the plastic constitutive equation to

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{3}{2}\frac{1}{\sigma_e}s_{ij} + 0.01\frac{\rho^*}{\rho_s}\delta_{ij}. \tag{7}$$

According to this model, we find

$$\dot{\hat{\sigma}} = \dot{\sigma}_e + 0.03\frac{\rho^*}{\rho_s}\dot{\sigma}_m. \tag{8}$$

The mean stress σ_m is incorporated into the model, and the effect of the foam material is represented by both the compressible plasticity, which is different from the dense material, and the relative density $\frac{\rho^*}{\rho_s}$, which reveals the effect of the size and geometry configuration of the cell. It is simple and can help us to construct some analytic solutions.

By using the above equations, the following deformation geometry relation

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right), \tag{9}$$

and the equilibrium equation

$$\frac{\partial \dot{\sigma}_{ij}}{\partial x_j} = 0, \tag{10}$$

where $\dot{\sigma}_{ij}$ is the stress rate, we can derive the solution of stress, deformation rate and velocity field of the metallic cellular materials according to appropriate boundary conditions. Since the continuum constitutive model is adopted, we can deduce analytic solutions, although they are more complicated than those derived from the classical rigid-hardening theory.

4 Stress field of the pure bending beam

For ideal isotropic rigid-hardening materials, in the case of plane strain,

$$\varepsilon_{zz} = 0, \tag{11}$$

the stress σ_{zz} normal to the x–y planes can be deduced from the flow theory as

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}), \tag{12}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are radial and circumferential stresses, respectively.

Since σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are principal stresses, substituting Eq. (12) into Eq. (1), Eq. (5a) with Eq. (2), then into Eq. (1), we can obtain the yield criterion for the present case

$$\left(0.01(1+\nu)\frac{\rho^*}{\rho_s}+1\right)\sigma_{rr} + \left(0.01(1+\nu)\frac{\rho^*}{\rho_s} - 1\right)\sigma_{\theta\theta} = \sigma^*_{pl}, \tag{13a}$$

as $a < r < c$, and

$$\left(0.01(1+\nu)\frac{\rho^*}{\rho_s} - 1\right)\sigma_{rr} + \left(0.01(1+\nu)\frac{\rho^*}{\rho_s} + 1\right)\sigma_{\theta\theta} = \sigma^*_{pl}, \tag{13b}$$

as $c < r < b$.

Due to the symmetry of the problem in the coordinate shown in Fig. 1, there is no shear stress, the equilibrium Eq. (10) takes the form

$$r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} = \sigma_{\theta\theta} - \sigma_{rr}, \tag{14}$$

and the boundary condition is

$$\sigma_{rr}(a,\theta) = 0, \qquad \sigma_{rr}(b,\theta) = 0. \tag{15}$$

In addition, at both ends of the beam there are St Venant boundary conditions, i.e., the conditions of total internal force and moment at unit thickness must be satisfied:

$$N = \int_a^b \sigma_{\theta\theta}\mathrm{d}r = 0, \qquad M = \int_a^b \sigma_{\theta\theta} r\,\mathrm{d}r. \tag{16}$$

From Eqs. (13a), (13b), (14) and boundary condition (15), we obtain immediately

$$\begin{aligned} \sigma_{rr} &= \frac{B}{A}\left[\left(\frac{r}{a}\right)^A - 1\right], \\ \sigma_{\theta\theta} &= B + \frac{(A+1)B}{A}\left[\left(\frac{r}{a}\right)^A - 1\right], \end{aligned} \tag{17a}$$

as $a < r < c$, and

$$\begin{aligned} \sigma_{rr} &= -\frac{B}{A}\left[\left(\frac{b}{r}\right)^A - 1\right], \\ \sigma_{\theta\theta} &= -B + \frac{(1-A)B}{A}\left[\left(\frac{b}{r}\right)^A - 1\right], \end{aligned} \tag{17b}$$

as $c < r < b$, where A and B are the combination of relevant material constants such as

$$A = \frac{0.02(1+\nu)\frac{\rho^*}{\rho_s}}{1 - 0.01(1+\nu)\rho^*/\rho_s}, \qquad B = \frac{\sigma^*_{pl}}{0.01(1+\nu)\rho^*/\rho_s - 1}, \tag{18}$$

this indicates that A and B are strongly dependent on the relative density ρ^*/ρ_s, which are constants for given material.

The radial stress decreases in magnitude with increasing radius, whereas the circumferential stress increases with increasing radius due to the yield criterion.

The radius of the neutral layer c is determined by the continuity condition of the radial stress at the elastic-plastic boundary

$$\sigma_{rr}(r \to c+0) = \sigma_{rr}(r \to c-0). \tag{19}$$

According to Eqs. (17a), (17b) and Eq. (19), one finds

$$c = \sqrt{ab}. \tag{20}$$

It is easy to check the first expression of the St Venant boundary condition (16). Substituting Eqs. (17b) and (20) into the second expression of Eq. (16) yields

$$M = \frac{2Ba^2}{4-A^2}\left(\frac{b}{a}\right)^{\frac{A+2}{2}} + \frac{Bb^2}{2(A-2)} - \frac{Ba^2}{2(A+2)}. \tag{21}$$

As an example, we conducted numerical calculation for the stress field of a beam with size $a = 0.1\,\text{m}$, $b = 0.3\,\text{m}$ of INCO nickel foam based on the above theoretical results. The material parameters are [12] $E = 0.271\,\text{GPa}$, $\sigma^*_{pl} = 0.811\,\text{MPa}$, $\nu = 0.3$.

The distributions of circumferential stress along a transverse section of the beam are shown in Fig. 2 for different ratios of ρ^*/ρ_s.

5 Velocity field in a pure bending beam

For the velocity field, it is quite different from the well-known Hill's solution [11] of the classical plasticity, here the problem is strongly dependent on the stress field, and is strongly rate-dependent as well. The determination of velocity field needs first a kinematics parameter. According to the characteristics of non-growing layer

$$(\theta + \mathrm{d}\theta)(r + \mathrm{d}r) = r\theta, \tag{22}$$

assume the angle of the bending beam growing steadily with a rate of $\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$. From Eq. (22) one finds that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{r}{\theta}\dot{\theta}. \tag{23}$$

The meaning of the steady angular growth may be described by the following mathematical expression

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{\theta}\left(\frac{\partial}{\partial\theta} - \frac{r}{\theta}\frac{\partial}{\partial r}\right). \tag{24}$$

Thus the stress rate and generalized effective stress rate can be expressed via Eq. (24) as

$$\dot{\sigma}_{rr} = \dot{\theta}\left(\frac{\partial}{\partial\theta} - \frac{r}{\theta}\frac{\partial}{\partial r}\right)\sigma_{rr}, \qquad \dot{\sigma}_{\theta\theta} = \dot{\theta}\left(\frac{\partial}{\partial\theta} - \frac{r}{\theta}\frac{\partial}{\partial r}\right)\sigma_{\theta\theta}, \tag{25}$$

and

$$\dot{\hat{\sigma}} = \dot{\theta}\left(\frac{\partial}{\partial\theta} - \frac{r}{\theta}\frac{\partial}{\partial r}\right)\hat{\sigma}. \tag{26}$$

We consider the region of $a < r < c$. Substituting Eq. (17a) into Eq. (25)

$$\dot{\sigma}_{rr} = -\dot{\theta}\frac{B}{\theta}\left(\frac{r}{a}\right)^A, \qquad \dot{\sigma}_{\theta\theta} = -\dot{\theta}\frac{(A+1)B}{\theta}\left(\frac{r}{a}\right)^A, \tag{27}$$

and then combining Eqs. (2) and (26) yields

$$\dot{\hat{\sigma}} = \dot{\theta}\frac{B}{\theta}\left[A - 0.01(A+2)(1+\nu)\frac{\rho*}{\rho_s}\right]\left(\frac{r}{a}\right)^A. \tag{28}$$

In view of Eqs. (6), (7) and (28)

$$\dot{\varepsilon}_{rr} = \frac{1}{\theta}\left[R_1\left(\frac{r}{a}\right)^A + R_2\right], \qquad \dot{\varepsilon}_{\theta\theta} = \frac{1}{\theta}\left[R_3\left(\frac{r}{a}\right)^A + R_2\right], \tag{29}$$

where

$$\begin{aligned} R_1 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[A - 0.01(A+2)(\nu+1)\frac{\rho^*}{\rho_s}\right] \times \left[\frac{A(1+\nu)-(1-2\nu)}{2A} + 0.01\frac{\rho^*}{\rho_s}\right], \\ R_2 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[A - 0.01(A+2)(\nu+1)\frac{\rho^*}{\rho_s}\right]\frac{1-2\nu}{2A}, \\ R_3 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[A - 0.01(A+2)(\nu+1)\frac{\rho^*}{\rho_s}\right] \times \left[\frac{A(\nu-2)-(1-2\nu)}{2A} + 0.01\frac{\rho^*}{\rho_s}\right]. \end{aligned} \tag{30}$$

The deformation geometry relation (9) can be written in a polar coordinate system like

$$\dot{\varepsilon}_{rr} = -\frac{\partial \dot{u}_r}{\partial r}, \qquad \dot{\varepsilon}_{\theta\theta} = -\frac{\partial \dot{u}_r}{\partial r} + \frac{1}{r}\frac{\partial \dot{u}_\theta}{\partial\theta}, \tag{31}$$

where $\dot{u}_r$ and $\dot{u}_\theta$ are the radial and circumferential components of the velocity field, respectively, and A and B are given by Eq. (18).

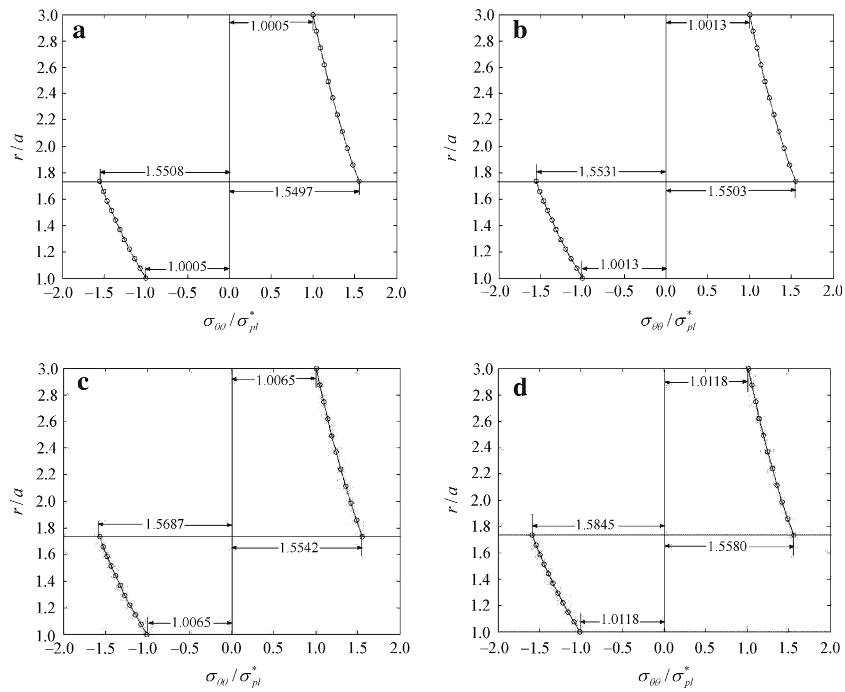

Fig. 2 The normalized circumferential stress verses normalized height of the beam for different ratios of ρ^*/ρ_s. **a** $\rho^*/\rho_s = 0.04$; **b** $\rho^*/\rho_s = 0.1$; **c** $\rho^*/\rho_s = 0.5$; **d** $\rho^*/\rho_s = 0.9$

In view of Eqs. (29) and (31), one can derive $\dot{u}_r$, which is then substituted into the second expression of Eq. (31) to obtain $\dot{u}_\theta$ by integrating it

$$\begin{aligned} \dot{u}_r &= -\left[\frac{aR_1}{A+1}\left(\frac{r}{a}\right)^{A+1} + R_2 r\right]\frac{1}{\theta}, \\ \dot{u}_\theta &= a\left(R_3 - \frac{R_1}{A+1}\right)\left(\frac{r}{a}\right)^{A+1}\ln\theta. \end{aligned} \tag{32a}$$

In similar manner, we can derive the velocity field in the region of $c < r < b$

$$\begin{aligned} \dot{u}_r &= \left[\frac{bR'_1}{A-1}\left(\frac{b}{r}\right)^{A-1} - R'_2 r\right]\frac{1}{\theta}, \\ \dot{u}_\theta &= a\left(R'_3 + \frac{R'_1}{A-1}\right)\left(\frac{b}{r}\right)^{A-1}\ln\theta, \end{aligned} \tag{32b}$$

in which

$$\begin{aligned} R'_1 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[0.01(2-A)(\nu+1)\frac{\rho^*}{\rho_s} - A\right] \\ &\quad \times \left[0.01\frac{\rho^*}{\rho_s} - \frac{A(1+\nu)+(1-2\nu)}{2A}\right], \\ R'_2 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[0.01(2-A)(\nu+1)\frac{\rho^*}{\rho_s} - A\right]\frac{1-2\nu}{2A}, \\ R'_3 &= \frac{\dot{\theta}B}{H(\hat{\sigma})}\left[0.01(2-A)(\nu+1)\frac{\rho^*}{\rho_s} - A\right] \\ &\quad \times \left[0.01\frac{\rho^*}{\rho_s} - \frac{A(\nu-2)+(1-2\nu)}{2A}\right]. \end{aligned} \tag{33}$$

Analysis of the velocity field in the classical plasticity theory depends upon the deformation geometry only, however, it does not hold for foam materials. For the latter, the determination of the velocity field depends explicitly upon the stress solution, and thus depends explicitly upon the constitutive law.

6 Discussion

Due to the presence of sub-structure—cells, metallic foams present the feature of compressible plasticity and thus lead to constitutive equations more complicated than that of incompressible plasticity which describes the behavior of fully dense solids.

The complexity of the constitutive law makes it difficult for solving boundary value problems of foam materials. Even if exact solutions are available in some cases, the incompressible condition holding true in the classical theory of plasticity cannot be used, and some new methods must be developed for the stress analysis of structures made of new materials. The present work is an exploration in this direction.

For pure bending beams as suggested by Hill but made of foam materials, the stress field can be statically determined and the results are obtained with some numerical illustrations shown in Fig. 2, where there are some similar features compared with results (e.g. the Hill's solution) given by classical plasticity. Since the present solutions are strongly dependent on the relative density ρ^*/ρ_s of the foam material, the differences between the two are distinct and understandable.

The solution of the velocity field is quite different from Hill's solution of classical plasticity, here the determination of velocity field is strongly dependent on the stress field, and the results are given by formulas (32a), (32b).

7 Conclusion

The TG model for the material is used to describe the behavior of a pure bending beam and an exact analytic solution is given, in which the effects of relative density of the materials on the stresses and deformation are revealed. Several examples for individual foams are discussed, showing the substantial importance of the cellular material's compressibility. A kinematics parameter is introduced into the analysis so that the velocity field can be determined. These insights may be helpful for developing structural stress analyses of the new material.

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