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RESEARCH PAPER

Nonlinear formulation for flexible multibody system with large deformation

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Abstract In this paper, nonlinear modeling for flexible multibody system with large deformation is investigated. Absolute nodal coordinates are employed to describe the displacement, and variational motion equations of a flexible body are derived on the basis of the geometric nonlinear theory, in which both the shear strain and the transverse normal strain are taken into account. By separating the inner and the boundary nodal coordinates, the motion equations of a flexible multibody system are assembled. The advantage of such formulation is that the constraint equations and the forward recursive equations become linear because the absolute nodal coordinates are used. A spatial double pendulum connected to the ground with a spherical joint is simulated to investigate the dynamic performance of flexible beams with large deformation. Finally, the resultant constant total energy validates the present formulation.

Keywords Flexible multibody system · Large deformation · Absolute nodal coordinate formulation

1 Introduction

The dynamic modeling theory of flexible multibody systems has been investigated for thirty years. The hybrid-coordinate formulation is based on identifying the configuration of each flexible body by means of two coordinate systems: the global fixed coordinate system and the body-fixed coordinate system. The advantage of this method is that the deformation of the flexible multibody system can be expressed by a small number of coordinates. Thus the rigid body motion and the elastic deformations are solved simultaneously [1]. However, ten years ago, it was shown that the conventional hybrid-coordinate formulation failed to capture the stiffening terms and provided the erroneous dynamic results in the case of high rotating speed [2]. Since then, the dynamic stiffening of flexible beams and plates has been investigated by many researches. To consider the geometric stiffening effect, Wallrapp et al. developed the motion equations of flexible multibody systems including stress stiffening effects [3,4]. The stress stiffness matrix was derived from the internal virtual work equations including nonlinear terms of the strain-displacement relationship and reference stresses induced by the existing loads before the deformation. The resultant stress stiffness matrix for a flexible body was obtained through quasi-static structural analyses. However, this formulation required a costly computation of the joint reaction forces for flexible multibody systems. A further improvement in the formulation was made by employing non-Cartesian deformation variables to obtain the motion equations for a slender beam or a thin plate [2,5–7,11]. The motion-induced stiffness term with the foreshortening deformation could account for the stiffening effect. Such method provided a simple expression of strain energy. It was as efficient as the conventional linear modeling method and as accurate as the nonlinear modeling methods. Moreover, there was no need to calculate the joint reaction forces for deriv-

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ing the stiffness terms. However, in these formulations, the generalized mass matrix and force matrix were highly nonlinear. In order to simplify the problem, the high order deformation terms in the mass and force matrices were neglected under the assumption of small deformation.

Berzeri et al. used absolute nodal coordinate formulation to investigate the dynamics of an elastic beam [8–10]. The nodal coordinates were defined in a global coordinate system, and the global slopes instead of angles were used to define the orientation of a flexible body. Such a method led to a simple form of the inertia forces. The centrifugal and Coriolis forces vanished from the motion equations. Furthermore, the expression of the strain energy was fully nonlinear and the nonlinear stiffness terms were naturally taken into account. Thus the absolute nodal coordinate formulation could be successfully used for the dynamic analysis of flexible beams and plates with large deformation. Recently, by combining the hybrid coordinate formulation and the absolute nodal coordinate formulation, not only the relative nodal displacement and the slope of each beam element with respect to the body-fixed frame of the central body but also the variables related to central body motion were included in the system generalized coordinate [12]. On the basis of the same precise strain-displacement relationship with the absolute nodal coordinate formulation, a new rigid-flexible coupling dynamic model for a spatial beam was established. In these publications, the dynamics of a single beam with large deformation was investigated.

In this paper, the absolute nodal coordinate formulation is extended to the flexible multibody system with large deformation. With both the shear strain and the transverse normal strain taken into consideration, the variational motion equations of a flexible body are derived on the basis of the geometric nonlinear theory. The motion equations for the flexible multibody system are assembled by separating the inner and the boundary nodal coordinates. The advantage of such formulation is that the constraint equations and the forward recursive equations become linear in the absolute coordinates. Finally, a spatial double pendulum connected to the ground with a spherical joint is simulated to display the dynamic performance of flexible beams with large deformation.

2 Description of displacement

A geometric nonlinear formulation of flexible multibody systems with large deformation is proposed. As shown in Fig. 1, $X_0Y_0Z_0$ represents the inertial coordinate system. Each body of the system $B_i\,(i = 1,\ldots,N)$ is divided into n elements and $X_eY_eZ_e$ is the element coordinate system of element $e(e = 1,\ldots,n)$. Let (x, y, z) be the coordinate of point k with respect to $X_eY_eZ_e$, and $(x_1, y_1, z_1),\ldots,(x_m, y_m, z_m)$ be the coordinates of the element nodes, by using absolute nodal coordinate formulation, the coordinate vector of the displacement of an arbitrary point of B_i is written as

$$\boldsymbol{r}(x, y, z, t) = \boldsymbol{S}(x, y, z)\boldsymbol{q}_e(t), \tag{1}$$

where $\boldsymbol{S}$ is the shape function matrix, and $\boldsymbol{q}_e$ represents the element nodal coordinates, which can be written as

$$\boldsymbol{q}_e = [\boldsymbol{e}_{11}^{\mathrm{T}} \quad \boldsymbol{e}_{12}^{\mathrm{T}} \quad \boldsymbol{e}_{13}^{\mathrm{T}} \quad \boldsymbol{e}_{14}^{\mathrm{T}} \cdots \boldsymbol{e}_{m1}^{\mathrm{T}} \quad \boldsymbol{e}_{m2}^{\mathrm{T}} \quad \boldsymbol{e}_{m3}^{\mathrm{T}} \quad \boldsymbol{e}_{m4}^{\mathrm{T}}]^{\mathrm{T}},$$

where the element nodal coordinates include the absolute displacements

$$\begin{aligned} \boldsymbol{e}_{11} &= \boldsymbol{r}|_{x=x_1,y=y_1,z=z_1}, \\ &\cdots, \\ \boldsymbol{e}_{m1} &= \boldsymbol{r}|_{x=x_m,y=y_m,z=z_m}, \end{aligned} \tag{2}$$

and the slopes of the element nodes

$$\begin{aligned} \boldsymbol{e}_{12} &= \partial\boldsymbol{r}/\partial x|_{x=x_1,y=y_1,z=z_1}, \\ \boldsymbol{e}_{13} &= \partial\boldsymbol{r}/\partial y|_{x=x_1,y=y_1,z=z_1}, \\ \boldsymbol{e}_{14} &= \partial\boldsymbol{r}/\partial z|_{x=x_1,y=y_1,z=z_1}, \\ &\cdots, \\ \boldsymbol{e}_{m2} &= \partial\boldsymbol{r}/\partial x|_{x=x_m,y=y_m,z=z_m}, \\ \boldsymbol{e}_{m3} &= \partial\boldsymbol{r}/\partial y|_{x=x_m,y=y_m,z=z_m}, \\ \boldsymbol{e}_{m4} &= \partial\boldsymbol{r}/\partial z|_{x=x_m,y=y_m,z=z_m}. \end{aligned} \tag{3}$$

The kinetic energy of element e reads

$$T_e = \frac{1}{2}\int_V \rho\dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\mathrm{d}V = \frac{1}{2}\dot{\boldsymbol{q}}_e^{\mathrm{T}}\boldsymbol{M}_e\dot{\boldsymbol{q}}_e, \tag{4}$$

where $\boldsymbol{M}_e$ is the element mass matrix, which is given by

$$\boldsymbol{M}_e = \int_V \rho\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}\mathrm{d}V. \tag{5}$$

It can be shown that by using global slopes instead of angles to define the orientation of a flexible body, the element mass matrix is constant. The virtual work of the gravitational force of the element is given by

$$\delta W_e = \int_V \delta\boldsymbol{r}^{\mathrm{T}}\boldsymbol{F}_g\mathrm{d}V = \delta\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{Q}_e, \tag{6}$$

where $\boldsymbol{F}_g = [F_{1g} \quad F_{2g} \quad F_{3g}]^{\mathrm{T}}$ represents the gravitational force vector defined in the inertial frame. The element generalized force vector takes the form

$$\boldsymbol{Q}_e = \int_V \boldsymbol{S}^{\mathrm{T}}\boldsymbol{F}_g\mathrm{d}V. \tag{7}$$

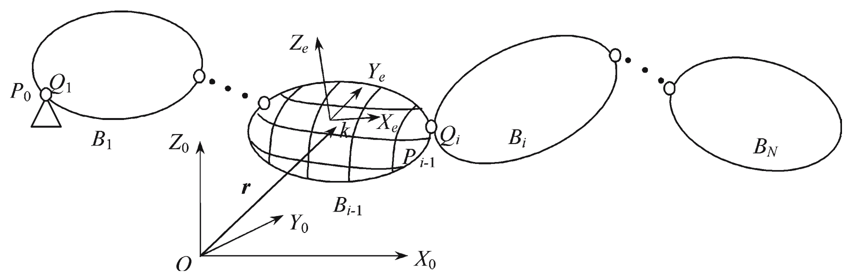

Fig. 1 Flexible multibody system with spherical constraints

It can be seen that by using absolute nodal coordinate formulation, the centrifugal and Coriolis forces vanish from the equations, and the generalized force vector is constant.

Let $\boldsymbol{q}$ denote the global nodal coordinate vector, and $\boldsymbol{B}_e$ denote the element Boolean matrix, one obtains that

$$\boldsymbol{q}_e = \boldsymbol{B}_e\boldsymbol{q}. \tag{8}$$

Thus, the kinetic energy and the generalized force vector of the beam are given by

$$\begin{aligned} T &= \sum_e T_e = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{q}}, \\ \delta W &= \sum_e \delta W_e = \delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}, \end{aligned} \tag{9}$$

where the generalized mass and force matrices of the beam are written as

$$\begin{aligned} \boldsymbol{M} &= \sum_e \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{M}_e\boldsymbol{B}_e, \\ \boldsymbol{Q} &= \sum_e \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{Q}_e. \end{aligned} \tag{10}$$

3 Nonlinear elastic force

On the basis of the nonlinear elastic theory, the strain tensor can be written as [10]

$$\boldsymbol{\varepsilon}_m = \frac{1}{2}(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J} - \boldsymbol{I}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{X}}\right) - \boldsymbol{I}\right), \tag{11}$$

where $\boldsymbol{I}$ is the unit matrix, and $\boldsymbol{J}$ is the derivative matrix of $\boldsymbol{r}$ with respect to $\boldsymbol{X} = [x \quad y \quad z]^{\mathrm{T}}$.

Denoting that $\boldsymbol{S}_x = \frac{\partial \boldsymbol{S}}{\partial x}, \boldsymbol{S}_y = \frac{\partial \boldsymbol{S}}{\partial y}, \boldsymbol{S}_z = \frac{\partial \boldsymbol{S}}{\partial z}$ and substituting Eq. (1) into Eq. (11), one obtains the following strain tensor

$$\boldsymbol{\varepsilon}_m = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{2}\left(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_a\boldsymbol{q}_e - 1\right), \\ \varepsilon_{22} &= \frac{1}{2}\left(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_b\boldsymbol{q}_e - 1\right), \end{aligned} \tag{13}$$

$$\begin{aligned} \varepsilon_{33} &= \frac{1}{2}(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_c\boldsymbol{q}_e - 1), \\ \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_d\boldsymbol{q}_e, \end{aligned} \tag{14}$$

$$\begin{aligned} \varepsilon_{13} = \varepsilon_{31} &= \frac{1}{2}\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_f\boldsymbol{q}_e, \\ \varepsilon_{23} = \varepsilon_{32} &= \frac{1}{2}\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_g\boldsymbol{q}_e, \end{aligned} \tag{15}$$

$$\begin{aligned} \boldsymbol{S}_a &= \boldsymbol{S}_x^{\mathrm{T}}\boldsymbol{S}_x, \\ \boldsymbol{S}_b &= \boldsymbol{S}_y^{\mathrm{T}}\boldsymbol{S}_y, \\ \boldsymbol{S}_c &= \boldsymbol{S}_z^{\mathrm{T}}\boldsymbol{S}_z, \end{aligned} \tag{16}$$

$$\begin{aligned} \boldsymbol{S}_d &= \boldsymbol{S}_x^{\mathrm{T}}\boldsymbol{S}_y, \\ \boldsymbol{S}_f &= \boldsymbol{S}_x^{\mathrm{T}}\boldsymbol{S}_z, \\ \boldsymbol{S}_g &= \boldsymbol{S}_y^{\mathrm{T}}\boldsymbol{S}_z. \end{aligned} \tag{17}$$

The stress and strain vectors then take the forms

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}, \tag{18}$$

$$\boldsymbol{\sigma} = [\,\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{21}\ \sigma_{13}\ \sigma_{31}\ \sigma_{23}\ \sigma_{32}\,], \tag{19}$$

$$\boldsymbol{\varepsilon} = [\,\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{21}\ \varepsilon_{13}\ \varepsilon_{31}\ \varepsilon_{23}\ \varepsilon_{32}\,], \tag{20}$$

where the matrix of elastic modulus is given by

$$\boldsymbol{E} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}, \tag{21}$$

In Eq. (21), λ and μ are given by

$$\begin{aligned} \lambda &= \frac{E\gamma}{(1+\gamma)(1-2\gamma)}, \\ \mu &= \frac{E}{2(1+\gamma)}, \end{aligned} \tag{22}$$

where E, γ represent elastic modulus and Poisson ratio, respectively.

The strain energy of the beam is given by

$$U=\sum_e \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\varepsilon}\mathrm{d}V=\sum_e \frac{1}{2}\int_V \big[(\lambda+2\mu)\varepsilon_{11}^2 + (\lambda+2\mu)\varepsilon_{22}^2+(\lambda+2\mu)\varepsilon_{33}^2\big]\mathrm{d}V + \sum_e \frac{1}{2}\int_V \big[2\lambda\varepsilon_{11}\varepsilon_{22}+2\lambda\varepsilon_{11}\varepsilon_{33}+2\lambda\varepsilon_{22}\varepsilon_{33} + 4\mu\varepsilon_{12}^2+4\mu\varepsilon_{13}^2+4\mu\varepsilon_{23}^2\big]\mathrm{d}V, \quad (23)$$

and the transpose matrix of the differentiation of U with respect to $\boldsymbol{q}$ reads

$$\left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^{\mathrm{T}}=\sum_e\int_V(\lambda+2\mu)\left[\varepsilon_{11}\left(\frac{\partial\varepsilon_{11}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}} +\varepsilon_{22}\left(\frac{\partial\varepsilon_{22}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}+\varepsilon_{33}\left(\frac{\partial\varepsilon_{33}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\right]\mathrm{d}V +\sum_e\int_V\lambda\left[\varepsilon_{11}\left(\frac{\partial\varepsilon_{22}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}+\varepsilon_{22}\left(\frac{\partial\varepsilon_{11}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\right]\mathrm{d}V +\sum_e\int_V\lambda\left[\varepsilon_{11}\left(\frac{\partial\varepsilon_{33}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}+\varepsilon_{33}\left(\frac{\partial\varepsilon_{11}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\right]\mathrm{d}V +\sum_e\int_V\lambda\left[\varepsilon_{22}\left(\frac{\partial\varepsilon_{33}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}+\varepsilon_{33}\left(\frac{\partial\varepsilon_{22}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\right]\mathrm{d}V +\sum_e\int_V 4\mu\left[\varepsilon_{12}\left(\frac{\partial\varepsilon_{12}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}+\varepsilon_{13}\left(\frac{\partial\varepsilon_{13}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}} +\varepsilon_{23}\left(\frac{\partial\varepsilon_{23}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\right]\mathrm{d}V. \quad (24)$$

Since $\boldsymbol{S}_a^{\mathrm{T}}=\boldsymbol{S}_a$, $\boldsymbol{S}_b^{\mathrm{T}}=\boldsymbol{S}_b$, $\boldsymbol{S}_c^{\mathrm{T}}=\boldsymbol{S}_c$, differentiating Eqs. (13)–(15) lead to

$$\begin{aligned}
\frac{\partial\varepsilon_{11}}{\partial\boldsymbol{q}}&=\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S}_a\boldsymbol{B}_e,\\
\frac{\partial\varepsilon_{22}}{\partial\boldsymbol{q}}&=\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S}_b\boldsymbol{B}_e,\\
\frac{\partial\varepsilon_{33}}{\partial\boldsymbol{q}}&=\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S}_c\boldsymbol{B}_e,\\
\frac{\partial\varepsilon_{12}}{\partial\boldsymbol{q}}&=\frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}(\boldsymbol{S}_d^{\mathrm{T}}+\boldsymbol{S}_d)\boldsymbol{B}_e,\\
\frac{\partial\varepsilon_{13}}{\partial\boldsymbol{q}}&=\frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}(\boldsymbol{S}_f^{\mathrm{T}}+\boldsymbol{S}_f)\boldsymbol{B}_e,\\
\frac{\partial\varepsilon_{23}}{\partial\boldsymbol{q}}&=\frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}_e^{\mathrm{T}}(\boldsymbol{S}_g^{\mathrm{T}}+\boldsymbol{S}_g)\boldsymbol{B}_e.
\end{aligned}\quad (25)$$

Substituting Eqs. (25) into Eq. (24), one obtains that

$$\left(\frac{\partial U}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}=(\boldsymbol{K}_1(\boldsymbol{q})-\boldsymbol{K}_2)\boldsymbol{q}, \quad (26)$$

where

$$\begin{aligned}
\boldsymbol{K}_1(\boldsymbol{q})&=\sum_e\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{K}_{1e}(\boldsymbol{q}_e)\boldsymbol{B}_e,\\
\boldsymbol{K}_2&=\sum_e\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{K}_{2e}\boldsymbol{B}_e,
\end{aligned}\quad (27)$$

$$\boldsymbol{K}_{1e}(\boldsymbol{q}_e)=(\lambda+2\mu)\boldsymbol{D}_1(\boldsymbol{q}_e)+\lambda\boldsymbol{D}_2(\boldsymbol{q}_e)+4\mu\boldsymbol{D}_3(\boldsymbol{q}_e), \quad (28)$$

$$\boldsymbol{K}_{2e}=(3\lambda+2\mu)\boldsymbol{D}_4, \quad (29)$$

$$\boldsymbol{D}_1(\boldsymbol{q}_e)=\frac{1}{2}\int_V\big[(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_a\boldsymbol{q}_e)\boldsymbol{S}_a+(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_b\boldsymbol{q}_e)\boldsymbol{S}_b +(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_c\boldsymbol{q}_e)\boldsymbol{S}_c\big]\mathrm{d}V, \quad (30)$$

$$\boldsymbol{D}_2(\boldsymbol{q}_e)=\frac{1}{2}\int_V\big[(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_a\boldsymbol{q}_e)\boldsymbol{S}_b+(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_b\boldsymbol{q}_e)\boldsymbol{S}_a\big]\mathrm{d}V +\frac{1}{2}\int_V\big[(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_a\boldsymbol{q}_e)\boldsymbol{S}_c+(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_c\boldsymbol{q}_e)\boldsymbol{S}_a\big]\mathrm{d}V +\frac{1}{2}\int_V\big[(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_c\boldsymbol{q}_e)\boldsymbol{S}_b+(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_b\boldsymbol{q}_e)\boldsymbol{S}_c\big]\mathrm{d}V, \quad (31)$$

$$\boldsymbol{D}_3(\boldsymbol{q}_e)=\frac{1}{4}\int_V(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_d\boldsymbol{q}_e)(\boldsymbol{S}_d+\boldsymbol{S}_d^{\mathrm{T}})\mathrm{d}V +\frac{1}{4}\int_V(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_f\boldsymbol{q}_e)(\boldsymbol{S}_f+\boldsymbol{S}_f^{\mathrm{T}})\mathrm{d}V +\frac{1}{4}\int_V(\boldsymbol{q}_e^{\mathrm{T}}\boldsymbol{S}_g\boldsymbol{q}_e)(\boldsymbol{S}_g+\boldsymbol{S}_g^{\mathrm{T}})\mathrm{d}V, \quad (32)$$

$$\boldsymbol{D}_4=\frac{1}{2}\int_V(\boldsymbol{S}_a+\boldsymbol{S}_b+\boldsymbol{S}_c)\mathrm{d}V. \quad (33)$$

By substituting Eqs. (13)–(15) into Eq. (23), the strain energy is given by

$$U=\sum_e U_e, \quad (34)$$

where

$$U_e=\frac{1}{4}\boldsymbol{q}_e^{\mathrm{T}}(\boldsymbol{K}_{1e}(\boldsymbol{q}_e)-2\boldsymbol{K}_{2e})\boldsymbol{q}_e+\frac{3}{8}(3\lambda+2\mu)V. \quad (35)$$

By referring to Eq. (26), variation of the strain energy is given by

$$\delta U=(\delta U)^{\mathrm{T}}=\delta\boldsymbol{q}^{\mathrm{T}}\left(\frac{\partial U}{\partial\boldsymbol{q}}\right)^{\mathrm{T}} =\delta\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{K}_1(\boldsymbol{q})-\boldsymbol{K}_2)\boldsymbol{q}. \quad (36)$$

Variational motion equations of a flexible body take the form

$$\delta\boldsymbol{q}\Big[-\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{Q}-\Big(\frac{\partial U}{\partial\boldsymbol{q}}\Big)^{\mathrm{T}}\Big]=0. \tag{37}$$

After substituting Eqs. (26) into Eq. (37), motion equations of B_i can be rewritten as

$$\delta\boldsymbol{q}[\boldsymbol{M}\ddot{\boldsymbol{q}}+(\boldsymbol{K}_1(\boldsymbol{q})-\boldsymbol{K}_2)\boldsymbol{q}-\boldsymbol{Q}]=0. \tag{38}$$

4 Equations of motion

Let $\boldsymbol{M}^{(i)}$, $\boldsymbol{K}^{(i)}$, $\boldsymbol{Q}^{(i)}$ and $\boldsymbol{q}^{(i)}$ represent the generalized mass matrix, stiffness matrix, force matrix and generalized coordinate vector of the flexible body B_i, respectively, variational motion equations of the flexible multibody system take the form

$$\sum_{i=1}^{N}\delta\boldsymbol{q}^{(i)\mathrm{T}}\big[\boldsymbol{M}^{(i)}\ddot{\boldsymbol{q}}^{(i)}+(\boldsymbol{K}_1^{(i)}(\boldsymbol{q}^{(i)})-\boldsymbol{K}_2^{(i)})\boldsymbol{q}^{(i)}-\boldsymbol{Q}^{(i)}\big]=0. \tag{39}$$

As shown in Fig. 1, B_{i-1} and B_i are connected with a spherical joint at nodes P_{i-1} and Q_i, and B_1 is connected to the ground with a spherical joint at node Q_1. Let $\boldsymbol{r}_P^{(i-1)}$ and $\boldsymbol{r}_Q^{(i)}$ denote the displacement vector of P_{i-1} and Q_i, the constraint equations read

$$\boldsymbol{r}_Q^{(i)}-\boldsymbol{r}_P^{(i-1)}=\boldsymbol{0},\quad i=1,\ldots,N. \tag{40}$$

By choosing adequate Boolean matrix, $\boldsymbol{q}^{(i)}$ can be written as $\boldsymbol{q}^{(i)}=[\,\boldsymbol{r}_Q^{(i)\mathrm{T}}\ \boldsymbol{q}_I^{(i)\mathrm{T}}\ \boldsymbol{r}_P^{(i)\mathrm{T}}\,]^{\mathrm{T}}$, where $\boldsymbol{q}_I^{(i)}$ represents the vector consisting of the displacements and slopes of the inner nodes and the slopes of the boundary nodes.

$\boldsymbol{q}^{(i)}$ can be written as

$$\boldsymbol{q}^{(i)}=\boldsymbol{C}_I^{(i)}\boldsymbol{q}_I^{(i)}+\boldsymbol{C}_B^{(i)}\boldsymbol{q}_B^{(i)},\quad i=1,\ldots,N, \tag{41}$$

$$\begin{aligned}&\boldsymbol{q}_B^{(i)}=\begin{bmatrix}\boldsymbol{q}_{B1}^{(i)}\\ \boldsymbol{q}_{B2}^{(i)}\end{bmatrix},\\ &\boldsymbol{q}_{B1}^{(i)}=\boldsymbol{r}_Q^{(i)},\\ &\boldsymbol{q}_{B2}^{(i)}=\boldsymbol{r}_P^{(i)},\end{aligned} \tag{42}$$

where

$$\begin{aligned}&\boldsymbol{C}_I^{(i)}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix},\\ &\boldsymbol{C}_B^{(i)}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}.\end{aligned} \tag{43}$$

Because P_0 and Q_1 are fixed to the ground, $\boldsymbol{q}_{B1}^{(1)}=\boldsymbol{r}_Q^{(1)}=\boldsymbol{r}_P^{(0)}$, therefore, $\boldsymbol{q}_{B1}^{(1)}$ is a constant matrix, which is not included in the generalized coordinates of the system. Considering the constraint Eq. (40), one obtains that $\boldsymbol{q}_{B1}^{(i)}=\boldsymbol{q}_{B2}^{(i-1)}$, $i=1,\ldots,N$. The independent generalized system coordinate vector $\boldsymbol{q}_g$ is given by

$$\boldsymbol{q}_g=\begin{bmatrix}\boldsymbol{q}_{gI}^{\mathrm{T}} & \boldsymbol{q}_{gB}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \tag{44}$$

where $\boldsymbol{q}_{gI}$ and $\boldsymbol{q}_{gB}$ represent the inner and generalized boundary coordinate vectors, respectively, which are written as

$$\begin{aligned}&\boldsymbol{q}_{gI}=\begin{bmatrix}\boldsymbol{q}_I^{(1)\mathrm{T}}\cdots\boldsymbol{q}_I^{(N)\mathrm{T}}\end{bmatrix}^{\mathrm{T}},\\ &\boldsymbol{q}_{gB}=\begin{bmatrix}\boldsymbol{q}_{B2}^{(1)\mathrm{T}}\cdots\boldsymbol{q}_{B2}^{(N)\mathrm{T}}\end{bmatrix}^{\mathrm{T}},\end{aligned} \tag{45}$$

and the relation between $\boldsymbol{q}_I^{(i)}$ and $\boldsymbol{q}_{gI}$, $\boldsymbol{q}_B^{(i)}$ and $\boldsymbol{q}_{gB}$ is given by

$$\boldsymbol{q}_I^{(1)}=\boldsymbol{a}^{(1)}\boldsymbol{q}_{gI},\quad \boldsymbol{q}_B^{(1)}=\boldsymbol{b}^{(0)}+\boldsymbol{b}^{(1)}\boldsymbol{q}_{gB}, \tag{46}$$

$$\boldsymbol{q}_I^{(i)}=\boldsymbol{a}^{(i)}\boldsymbol{q}_{gI},\quad \boldsymbol{q}_B^{(i)}=\boldsymbol{b}^{(i)}\boldsymbol{q}_{gB},\quad i>1, \tag{47}$$

where

$$\begin{aligned}&\boldsymbol{a}^{(i)}=\begin{bmatrix}\boldsymbol{a}_1^{(i)}\cdots\boldsymbol{a}_N^{(i)}\end{bmatrix},\\ &\boldsymbol{a}_k^{(i)}=\begin{cases}\boldsymbol{I}, & k=i,\\ \boldsymbol{0}, & k\neq i,\end{cases}\\ &\boldsymbol{b}^{(0)}=\begin{bmatrix}\boldsymbol{r}_P^{(0)}\\ \boldsymbol{0}\end{bmatrix},\\ &\boldsymbol{b}^{(i)}=\begin{bmatrix}\boldsymbol{b}_1^{(i)}\\ \boldsymbol{b}_2^{(i)}\end{bmatrix},\ i=1,\ldots,N,\\ &\boldsymbol{b}_1^{(i)}=\begin{bmatrix}\boldsymbol{b}_{11}^{(i)}\cdots\boldsymbol{b}_{1N}^{(i)}\end{bmatrix},\\ &\boldsymbol{b}_2^{(i)}=\begin{bmatrix}\boldsymbol{b}_{21}^{(i)}\cdots\boldsymbol{b}_{2N}^{(i)}\end{bmatrix},\\ &\boldsymbol{b}_{1k}^{(1)}=\boldsymbol{0},\\ &\boldsymbol{b}_{1k}^{(i)}=\begin{cases}\boldsymbol{I}, & k=i-1,\ i>1,\\ \boldsymbol{0}, & k\neq i-1,\end{cases}\\ &\boldsymbol{b}_{2k}^{(i)}=\begin{cases}\boldsymbol{I}, & k=i,\\ \boldsymbol{0}, & k\neq i.\end{cases}\end{aligned}$$

From Eqs. (44), (46) and (47), $\boldsymbol{q}_I^{(i)}$ and $\boldsymbol{q}_B^{(i)}$ can be written as

$$\boldsymbol{q}_I^{(1)}=\boldsymbol{A}^{(1)}\boldsymbol{q}_g,\quad \boldsymbol{q}_B^{(1)}=\boldsymbol{b}^{(0)}+\boldsymbol{B}^{(1)}\boldsymbol{q}_g, \tag{48}$$

$$\boldsymbol{q}_I^{(i)}=\boldsymbol{A}^{(i)}\boldsymbol{q}_g,\quad \boldsymbol{q}_B^{(i)}=\boldsymbol{B}^{(i)}\boldsymbol{q}_g,\quad i>1, \tag{49}$$

where

$$\boldsymbol{A}^{(i)}=\begin{bmatrix}\boldsymbol{a}^{(i)} & \boldsymbol{0}\end{bmatrix},\quad \boldsymbol{B}^{(i)}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{b}^{(i)}\end{bmatrix}. \tag{50}$$

Substitution of Eqs. (48) and (49) into (41) yields

$$\boldsymbol{q}^{(1)} = \boldsymbol{C}_B^{(1)}\boldsymbol{b}^{(0)} + \left(\boldsymbol{C}_I^{(1)}\boldsymbol{A}^{(1)} + \boldsymbol{C}_B^{(1)}\boldsymbol{B}^{(1)}\right)\boldsymbol{q}_g, \tag{51}$$

$$\boldsymbol{q}^{(i)} = \left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right)\boldsymbol{q}_g, \quad i > 1. \tag{52}$$

By using Eqs. (51) and (52), variational Eqs. (39) read

$$\delta\boldsymbol{q}_g^{\mathrm{T}}\left(\boldsymbol{M}_g\ddot{\boldsymbol{q}}_g + \boldsymbol{K}_g(\boldsymbol{q}_g)\boldsymbol{q}_g - \boldsymbol{Q}_g\right) = \boldsymbol{0}, \tag{53}$$

where

$$\boldsymbol{M}_g = \sum_{i=1}^{N}\left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right)^{\mathrm{T}}\boldsymbol{M}^{(i)} \times \left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right), \tag{54}$$

$$\boldsymbol{K}_g(\boldsymbol{q}_g) = \sum_{i=1}^{N}\left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right)^{\mathrm{T}} \times \left(\boldsymbol{K}_1^{(i)}(\boldsymbol{q}^{(i)}) - \boldsymbol{K}_2^{(i)}\right)\left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right), \tag{55}$$

$$\boldsymbol{Q}_g = \sum_{i=1}^{N}\left(\boldsymbol{C}_I^{(i)}\boldsymbol{A}^{(i)} + \boldsymbol{C}_B^{(i)}\boldsymbol{B}^{(i)}\right)^{\mathrm{T}}\boldsymbol{Q}^{(i)} - \left(\boldsymbol{C}_I^{(1)}\boldsymbol{A}^{(1)} + \boldsymbol{C}_B^{(1)}\boldsymbol{B}^{(1)}\right)^{\mathrm{T}} \times \left(\boldsymbol{K}_1^{(1)}(\boldsymbol{q}^{(1)}) - \boldsymbol{K}_2^{(1)}\right)\boldsymbol{C}_B^{(1)}\boldsymbol{b}^{(0)}. \tag{56}$$

Since $\boldsymbol{q}_g$ is an independent generalized coordinate vector, the motion equations of the flexible multibody system take the form

$$\boldsymbol{M}_g\ddot{\boldsymbol{q}}_g + \boldsymbol{K}_g(\boldsymbol{q}_g)\boldsymbol{q}_g = \boldsymbol{Q}_g. \tag{57}$$

5 Calculation of constraint force

For B_i, the virtual work of the constraint force is given by

$$\delta W_R = \delta\boldsymbol{r}_Q^{(i)\mathrm{T}}\boldsymbol{R}_1^{(i)} + \delta\boldsymbol{r}_P^{(i)\mathrm{T}}\boldsymbol{R}_2^{(i)}, \tag{58}$$

where $\boldsymbol{R}_1^{(i)}, \boldsymbol{R}_2^{(i)}, 1 \le i \le N-1$ represent the constraint force coordinate vectors at point Q_i and P_i, respectively. Then, the motion equations of B_i are given by

$$\boldsymbol{M}^{(i)}\ddot{\boldsymbol{q}}^{(i)} + \left(\boldsymbol{K}_1^{(i)}(\boldsymbol{q}^{(i)}) - \boldsymbol{K}_2^{(i)}\right)\boldsymbol{q}^{(i)} = \boldsymbol{Q}^{(i)} + \boldsymbol{R}^{(i)}, \tag{59}$$

where $\boldsymbol{q}^{(i)} = \begin{bmatrix}\boldsymbol{r}_Q^{(i)\mathrm{T}} & \boldsymbol{q}_I^{(i)\mathrm{T}} & \boldsymbol{r}_P^{(i)\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, and the generalized constraint force vector $\boldsymbol{R}^{(i)}$ is written as

$$\boldsymbol{R}^{(i)} = \begin{bmatrix}\boldsymbol{R}_1^{(i)\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{R}_2^{(i)\mathrm{T}}\end{bmatrix}^{\mathrm{T}}. \tag{60}$$

In the case $i = N, \boldsymbol{R}_2^{(i)} = \boldsymbol{0}, \boldsymbol{R}^{(i)} = \begin{bmatrix}\boldsymbol{R}_1^{(i)\mathrm{T}} \ \boldsymbol{0}^{\mathrm{T}} \ \boldsymbol{0}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$.

Firstly, the ordinary differential Eq. (57) are solved, and then, $\boldsymbol{R}_1^{(i)}, \boldsymbol{R}_2^{(i)}$ can be calculated from Eq. (59).

6 Simulation

In this section, a spatial double pendulum connected to the ground with a spherical joint is simulated. The properties for each rectangular beam are given as follows:

Length, $l = 1.2$ m, width, $b = 0.01$ m, height, $h = 0.01$ m, mass density, $\rho = 5{,}540\,\mathrm{kg/m^3}$, elastic modulus, $E = 1 \times 10^7\,\mathrm{N/m^2}$, Poisson ratio $\gamma = 0.3$, respectively.

The beam is divided into 6 elements. For each element with two nodes, the shape function matrix is given by

$$\boldsymbol{S} = \begin{bmatrix}\boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} & \boldsymbol{S}_{14} & \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} & \boldsymbol{S}_{24}\end{bmatrix}^{\mathrm{T}}, \tag{61}$$

where

$$\boldsymbol{S}_{11} = [1 - 3(x/l)^2 + 2(x/l)^3]\boldsymbol{I}_{3\times3}, \quad \boldsymbol{S}_{12} = [x/l - 2(x/l)^2 + (x/l)^3]\boldsymbol{I}_{3\times3}, \tag{62}$$

$$\boldsymbol{S}_{13} = (1 - x/l)(y/b)\boldsymbol{I}_{3\times3}, \quad \boldsymbol{S}_{14} = (1 - x/l)(z/h)\boldsymbol{I}_{3\times3}, \tag{63}$$

$$\boldsymbol{S}_{21} = [3(x/l)^2 - 2(x/l)^3]\boldsymbol{I}_{3\times3}, \quad \boldsymbol{S}_{22} = [-(x/l)^2 + (x/l)^3]\boldsymbol{I}_{3\times3}, \tag{64}$$

$$\boldsymbol{S}_{23} = (x/l)(y/b)\boldsymbol{I}_{3\times3}, \quad \boldsymbol{S}_{24} = (x/l)(z/h)\boldsymbol{I}_{3\times3}. \tag{65}$$

The element nodal coordinates $\boldsymbol{q}_e$ can be written as

$$\boldsymbol{q}_e = \begin{bmatrix}\boldsymbol{e}_{11}^{\mathrm{T}} & \boldsymbol{e}_{12}^{\mathrm{T}} & \boldsymbol{e}_{13}^{\mathrm{T}} & \boldsymbol{e}_{14}^{\mathrm{T}} & \boldsymbol{e}_{21}^{\mathrm{T}} & \boldsymbol{e}_{22}^{\mathrm{T}} & \boldsymbol{e}_{23}^{\mathrm{T}} & \boldsymbol{e}_{24}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \tag{66}$$

where the element nodal coordinates include the absolute displacements

$$\boldsymbol{e}_{11} = \boldsymbol{r}|_{x=0,y=0,z=0}, \quad \boldsymbol{e}_{21} = \boldsymbol{r}|_{x=l,y=0,z=0}, \tag{67}$$

and the slopes of the element nodes

$$\boldsymbol{e}_{12} = \partial\boldsymbol{r}/\partial x|_{x=0,y=0,z=0}, \quad \boldsymbol{e}_{13} = \partial\boldsymbol{r}/\partial y|_{x=0,y=0,z=0}, \quad \boldsymbol{e}_{14} = \partial\boldsymbol{r}/\partial z|_{x=0,y=0,z=0}, \tag{68}$$

$$\boldsymbol{e}_{22} = \partial\boldsymbol{r}/\partial x|_{x=l,y=0,z=0}, \quad \boldsymbol{e}_{23} = \partial\boldsymbol{r}/\partial y|_{x=l,y=0,z=0}, \quad \boldsymbol{e}_{24} = \partial\boldsymbol{r}/\partial z|_{x=l,y=0,z=0}. \tag{69}$$

In order to calculate the deformation variables, a body-fixed coordinate system must be used. Cantilevered-free boundary conditions are applied to each beam. For B_i $(i = 1, 2)$, the origin of the body-fixed coordinate system is located at point O_i. Let $\boldsymbol{e}_1^{(i)}$ be the coordinate unit

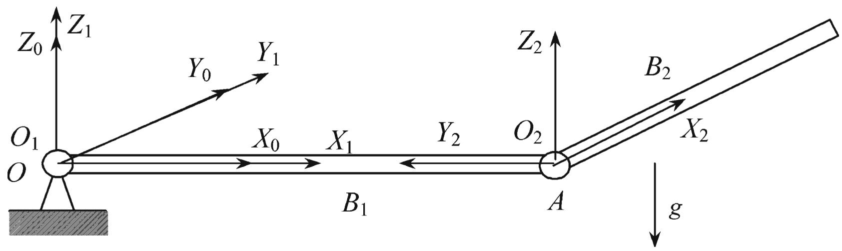

Fig. 2 Spatial double pendulum with large deformation

vector along the tangent of the deformed neutral axis of B_i, $\boldsymbol{e}_1^{(i)}$ can be written as [10]

$$\boldsymbol{e}_1^{(i)} = \frac{\partial \boldsymbol{r}^{(i)}/\partial x}{|\partial \boldsymbol{r}^{(i)}/\partial x|}, \tag{70}$$

where $|\partial \boldsymbol{r}^{(i)}/\partial x| = \sqrt{(\partial \boldsymbol{r}^{(i)}/\partial x)^{\mathrm{T}}(\partial \boldsymbol{r}^{(i)}/\partial x)}$.

Let $\boldsymbol{b}$ be the coordinate unit vector along the tangent of the deformed Y_i axis. $\boldsymbol{b}$ can be written as

$$\boldsymbol{b} = \frac{\partial \boldsymbol{r}^{(i)}/\partial y}{|\partial \boldsymbol{r}^{(i)}/\partial y|}, \tag{71}$$

where

$$|\partial \boldsymbol{r}^{(i)}/\partial y| = \sqrt{\left(\partial \boldsymbol{r}^{(i)}/\partial y\right)^{\mathrm{T}}\left(\partial \boldsymbol{r}^{(i)}/\partial y\right)}. \tag{72}$$

Due to the shear deformation, $\boldsymbol{b}$ is not perpendicular to $\boldsymbol{e}_1^{(i)}$, therefore, $\boldsymbol{e}_2^{(i)}$ is not exactly collinear with $\boldsymbol{b}$ in spite that the deviation angle is very small. In order to obtain $\boldsymbol{e}_2^{(i)}$ and $\boldsymbol{e}_3^{(i)}$, which satisfy $\boldsymbol{e}_3^{(i)} = \boldsymbol{e}_1^{(i)} \times \boldsymbol{e}_2^{(i)}$, firstly $\boldsymbol{e}_3^{(i)}$ is defined as

$$\boldsymbol{e}_3^{(i)} = \frac{\boldsymbol{e}_1^{(i)} \times \boldsymbol{b}}{|\boldsymbol{e}_1^{(i)} \times \boldsymbol{b}|}, \tag{73}$$

and then $\boldsymbol{e}_2^{(i)}$ can be obtained from the relation

$$\boldsymbol{e}_2^{(i)} = \boldsymbol{e}_3^{(i)} \times \boldsymbol{e}_1^{(i)}.$$

$\boldsymbol{e}_2^{(i)}$ and $\boldsymbol{e}_3^{(i)}$ can be written as

$$\boldsymbol{e}_3^{(i)} = \frac{\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}}{|\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}|}, \qquad \boldsymbol{e}_2^{(i)} = \tilde{\boldsymbol{e}}_3^{(i)}\boldsymbol{e}_1^{(i)}, \tag{74}$$

where $\tilde{\boldsymbol{e}}_1^{(i)}$ and $\tilde{\boldsymbol{e}}_3^{(i)}$ represent the skew-symmetric matrices corresponding to the coordinate vector $\boldsymbol{e}_1^{(i)}$ and $\boldsymbol{e}_3^{(i)}$, and $|\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}|$ is given by

$$|\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}| = \sqrt{\left(\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}\right)^{\mathrm{T}}\left(\tilde{\boldsymbol{e}}_1^{(i)}\boldsymbol{b}\right)}. \tag{75}$$

Let $\boldsymbol{i}_i, \boldsymbol{j}_i, \boldsymbol{k}_i$ be the unit vectors of the body-fixed frame along X_i, Y_i, Z_i axes, respectively, $\boldsymbol{i}_i, \boldsymbol{j}_i, \boldsymbol{k}_i$ can be written as

$$\boldsymbol{i}_i = \boldsymbol{e}_1^{(i)}|_{x=0}, \qquad \boldsymbol{j}_i = \boldsymbol{e}_2^{(i)}|_{x=0}, \qquad \boldsymbol{k}_i = \boldsymbol{e}_3^{(i)}|_{x=0}. \tag{76}$$

The transformation matrix $\boldsymbol{A}_i$ of $O_iX_iY_iZ_i$ with respect to the global coordinate system $O_0X_0Y_0Z_0$ can be written as

$$\boldsymbol{A}_i = \left[\, \boldsymbol{i}_i\, \boldsymbol{j}_i\, \boldsymbol{k}_i \,\right], \tag{77}$$

and by using the expression

$$\boldsymbol{r}^{(i)}(l_i) = \boldsymbol{r}^{(i)}(0) + \boldsymbol{A}_i([\, l_i\ 0\ 0\,]^{\mathrm{T}} + \boldsymbol{u}^{(i)}), \tag{78}$$

the deformation coordinate vector of the tip point is given by

$$\begin{aligned} \boldsymbol{u}^{(i)} &= [u_x^{(i)}\ u_y^{(i)}\ u_z^{(i)}]^{\mathrm{T}} \\ &= \boldsymbol{A}_i^{\mathrm{T}}[\boldsymbol{r}^{(i)}(l_i) - \boldsymbol{r}^{(i)}(0)] - [l_i\ 0\ 0]^{\mathrm{T}}. \end{aligned} \tag{79}$$

Initially, two beams are located in a horizontal plane, and the neutral axis of B_2 is perpendicular to that of B_1, as shown in Fig. 2. The beams are in a static state and are not deformed. The system motion is caused by gravitational force in negative Z_0 direction. Runge Kutta integration method is employed in the simulation in which the time step size is 10^{-5} s and the error tolerance is set to be 10^{-8} m.

The constraint forces exerted on B_2 for $E = 1 \times 10^6\,\mathrm{N/m^2}$ and $E = 1 \times 10^7\,\mathrm{N/m^2}$ are shown in Figs. 3 and 4. It can be seen that high frequency vibration of R_x and R_y for $E = 1 \times 10^7\,\mathrm{N/m^2}$ is excited. The reason is that with the increase of elastic modulus, the frequency of longitudinal vibration increases rapidly, thus, high frequency vibration of the constraint forces in X_0 and Y_0 direction is induced. Therefore, the vibration frequency of the constraint forces decreases with decreasing elastic modulus. The system energy is shown in Figs. 5 and 6. It is interesting to notice that with decreasing elastic modulus, the strain energy increases significantly, which leads to the decrease of gravitational energy. It can be seen that the difference between the kinetic energy for different elastic modulus is small. However, the difference is rather distinct at the peak point for $t = 0.75$ s due to the sudden increase of the strain energy. The resultant constant total energy validates the present formulation.

The deformations of the tip points of the inner and outer beams in X_i, Y_i and Z_i directions are shown in Figs. 7, 8, 9, 10. The deformations in each direction for $E = 1 \times 10^6\,\mathrm{N/m^2}$ is larger than those for

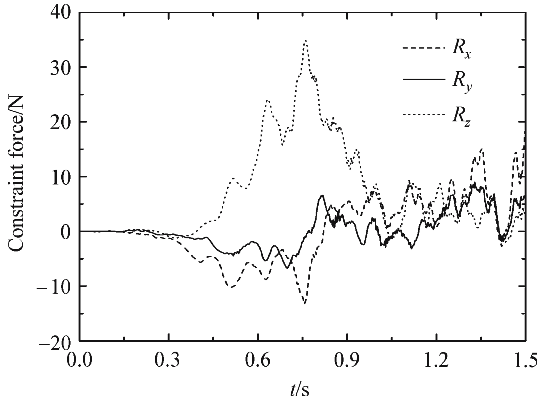

Fig. 3 Constraint force exerted on B_2 ($E = 1 \times 10^6$ N/m$^2 z$)

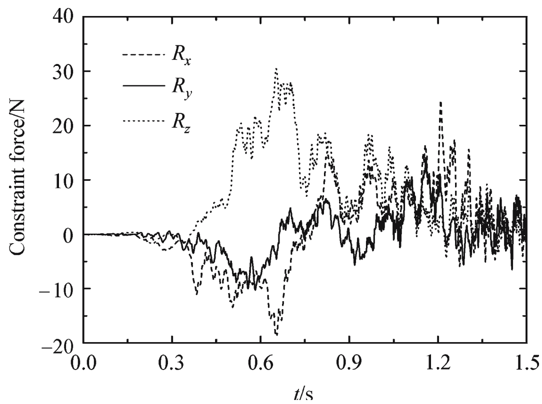

Fig. 4 Constraint force exerted on B_2 ($E = 1 \times 10^7$ N/m^2)

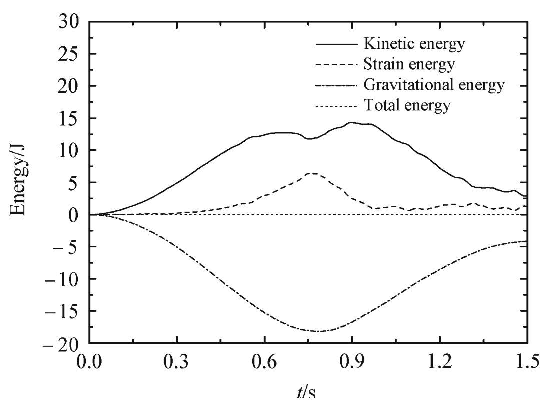

Fig. 5 System energy ($E = 1 \times 10^6$ N/m^2)

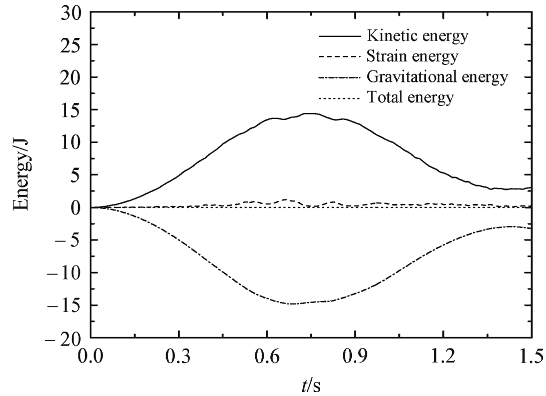

Fig. 6 System energy ($E = 1 \times 10^7$ N/m^2)

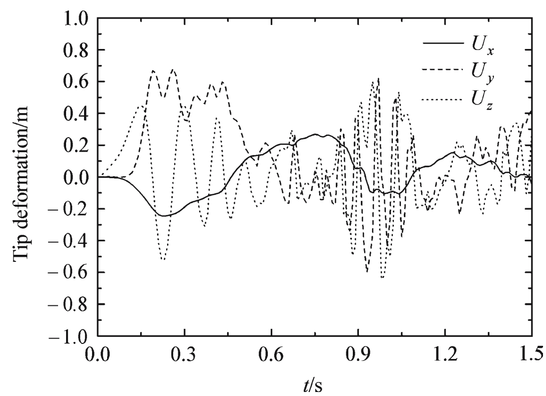

Fig. 7 Tip deformation of the inner beam ($E = 1 \times 10^6$ N/m^2)

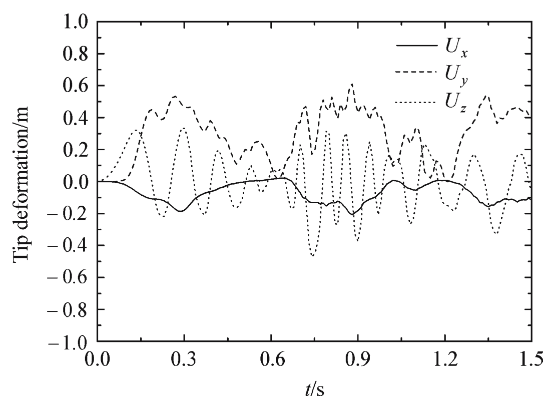

Fig. 8 Tip deformation of the inner beam ($E = 1 \times 10^7$ N/m^2)

$E = 1 \times 10^7$ N/m^2. For $E = 1 \times 10^6$ N/m^2, the maximum value of the tip lateral deformations of the two beams in Y_i and Z_i directions exceeds 0.8 m, which is three quarters of the beam length. Therefore it can be considered as a large deformation problem. It is interesting to notice that due to the coupling of the longitudinal and lateral deformations, the longitudinal deformation in X_i direction is significantly influenced by the large deformations in Y_i and Z_i directions. The tip longitudinal deformation of the outer beam is negative due to the foreshortening effect and the absolute value of u_x increases rapidly when the absolute values of the deformations in Y_i and Z_i directions reach their peaks. For $E = 1 \times 10^7$ N/m^2, the tip longitudinal deformation of the inner beam is negative, and for $E = 1 \times 10^6$ N/m^2, the tip longitudinal deformation of the inner beam is positive due to the significant increase of the axial stretch. In the case of

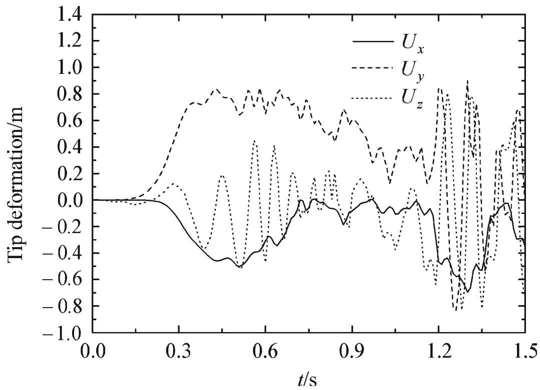

Fig. 9 Tip deformation of the outer beam ($E = 1 \times 10^6$ N/m^2)

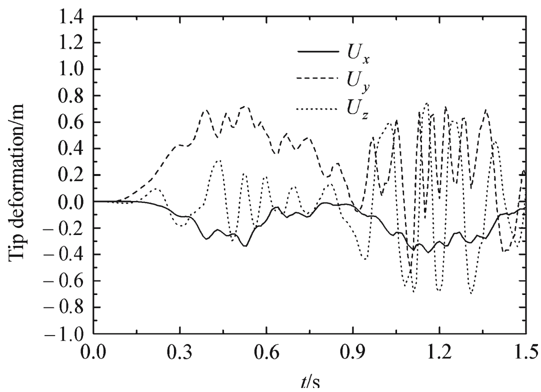

Fig. 10 Tip deformation of the outer beam ($E = 1 \times 10^7$ N/m^2)

$E = 1 \times 10^6$ N/m^2, the stretch effect is more significant than the foreshortening effect.

7 Conclusions

The absolute nodal coordinate formulation is extended to the flexible multibody systems with large deformation in this paper. Conclusions are drawn as follows:

With the decrease of the elastic modulus, the vibration amplitude of the lateral deformations in Y_i and Z_i directions increases significantly. Due to the coupling of the longitudinal and lateral deformations, the absolute value of the longitudinal deformation increases rapidly when the amplitudes of the deformations in Y_i and Z_i directions reach their peaks. Furthermore, with decreasing elastic modulus, the effect of the large deformation on the kinetic and gravitational energy becomes more prominent, whereas the vibration frequency of the constraint forces decreases.

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