非平整运动海底上 n 层流动中波浪传播的 Hamilton 逼近[1)]

黄 虎
(上海大学, 上海市应用数学和力学研究所, 上海 200072)
(中国科学院力学研究所非线性力学国家重点实验室, 北京 100080)

摘要 在海洋水域, 界面波对大尺度变化流的作用是一种典型的分层流动现象. 考虑一不可压缩、无黏的分层势流运动, 建立了一个在非平整运动海底上的 n 层流体演化系统, 并对其进行了 Hamilton 描述. 每层流体具有各自的常密度、均匀流水平速度, 其厚度由未扰动和扰动部分构成. 相对于顶层流体的自由表面, 刚性、运动的海底具有一般地形变化特征. 在明确指出 n 层流体运动的控制方程和各层交界面上的运动学、动力学边界条件 (包含各层交界面上张力效应) 后, 对该分层流动力系统进行了 Hamilton 构造, 即给出其正则方程和其下述的正则变量: 各交界面位移和各交界面上的动量势密度差.

关键词 Hamilton 描述, 界面波, 大尺度流, n 层流体, 非平整运动海底

引 言

无论是从理论还是从实践的观点来看, Hamilton 流体动力学极具深层次的意境, 例如现代孤子理论主要凭借控制方程的 Hamilton 结构而得以建立. 相对于单层流动的自由表面波运动及其日渐完善的 Hamilton 描述 [1,2], 分层流运动呈现出更为丰富的时空结构演化特性, 有待于采用 Hamilton 的观点和方法对其进行广泛而深入的探索和认识 [3~6].

分层流就其本质而言是连续分层现象, 与海况条件密切相关, 海底地形的变化可以直接影响它的动力学特性. 最为简单的每层为常量密度的两层流系统, 可以产生一种波动模式. 推而广之, 密度分别为常量的 n 层流体系统, 则能派生出 $n-1$ 种波动模式, 这是对连续分层流的一种精确逼近, 可以充分反映界面波对大尺度变化流的显著作用. 最近, 黄虎 [7,8] 给出了两层流、 n 层流系统的波动和波 - 流相互作用的 Hamilton 描述. 本文在此基础上, 考虑近岸非平整海底地形的影响, 并刻画运动海底变化的效应, 以有效描述相应的 Hamilton 逼近.

1 控制方程和边界条件

考虑一个带有运动海底边界的 n 层流体系统, 如图 1 所示. 第 i 层的密度、压力和厚度分别为 ρ_i, p_i 和 d_i, 其未扰动的厚度和均匀水平速度分别为 $\bar{d}_i$ 和 U_i. 于是, 第 i 层的上交界面位移可表示为

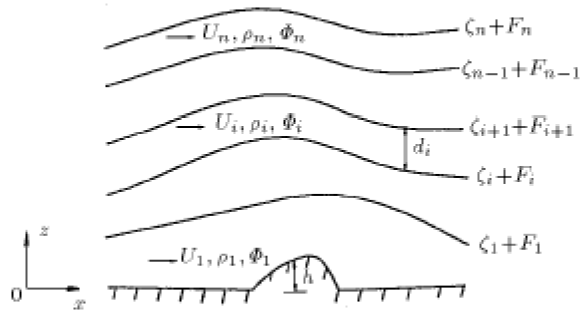

图 1 n 层流体坐标系统

Fig.1 A sketch of the coordinate system for a n-layered fluid

$$z=\sum_{j=1}^{i} d_j = F_i+\zeta_i\,,\quad F_i=\sum_{j=1}^{i}\bar{d}_j \tag{1}$$

假定流动为无黏、不可压缩, 则每层为无旋运动, 其速度势 Φ_i 可写为 [9]

$$\left.\begin{array}{c}\Phi_i(\boldsymbol{x},z,t)=U_i\cdot\boldsymbol{x}-\left(\dfrac{1}{2}|U_i|^2+A\right)t+\phi_i\\ i=1,2,\cdots,n\end{array}\right\} \tag{2}$$

其中, $\boldsymbol{x}\equiv(x,y)$ 表示水平坐标, A 为二阶 Stokes 波的近似常数, $\phi_i(\boldsymbol{x},z,t)$ 可被视为界面波扰动势.

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由此可给出 n 层流体运动的控制方程和在交界面及海底处的运动学、动力学边界条件

$$\nabla^2\phi_i+\frac{\partial^2\phi_i}{\partial z^2}=0,\qquad i=1,2,\cdots,n \tag{3}$$

$$\frac{\partial\zeta_n}{\partial t}=\frac{\partial\phi_n}{\partial z}-\nabla\zeta_n\cdot(U_n+\nabla\phi_n),\quad z=\zeta_n+F_n \tag{4a}$$

$$\left.\begin{aligned}&\frac{\partial\zeta_i}{\partial t}=\frac{\partial\phi_i}{\partial z}-\nabla\zeta_i\cdot(U_i+\nabla\phi_i)=\frac{\partial\phi_{i+1}}{\partial z}-\\&\quad\nabla\zeta_i\cdot(U_{i+1}+\nabla\phi_{i+1})\\&\quad z=\zeta_i+F_i\,,\ i=1,2,\cdots,n-1\end{aligned}\right\} \tag{4b}$$

$$\frac{\partial h}{\partial t}=\frac{\partial\phi_1}{\partial z}-\nabla h\cdot(U_1+\nabla\phi_1),\quad z=h \tag{4c}$$

$$\begin{aligned}&\rho_n\Big\{\frac{\partial\phi_n}{\partial t}-\frac{1}{2}|U_n|^2-A+\frac{1}{2}\Big[(U_n+\nabla\phi_n)^2+\\&\Big(\frac{\partial\phi_n}{\partial z}\Big)^2\Big]+g(\zeta_n+F_n)\Big\}-\rho_n g\bar{d}_n=\\&\tau_n\nabla\cdot\Big[\frac{\nabla\zeta_n}{\sqrt{1+(\nabla\zeta_n)^2}}\Big],\quad z=\zeta_n+F_n\end{aligned} \tag{5a}$$

$$\left.\begin{aligned}&\rho_i\Big\{\frac{\partial\phi_i}{\partial t}-\frac{1}{2}|U_i|^2-A+\frac{1}{2}\Big[(U_i+\nabla\phi_i)^2+\Big(\frac{\partial\phi_i}{\partial z}\Big)^2\Big]+\\&g(\zeta_i+F_i)\Big\}-\rho_{i+1}\Big\{\frac{\partial\phi_{i+1}}{\partial t}-\frac{1}{2}|U_{i+1}|^2-A+\\&\frac{1}{2}\Big[(U_{i+1}+\nabla\phi_{i+1})^2+\Big(\frac{\partial\phi_{i+1}}{\partial z}\Big)^2\Big]+g(\zeta_i+F_i)\Big\}=\\&\tau_i\nabla\cdot\Big[\frac{\nabla\zeta_i}{\sqrt{1+(\nabla\zeta_i)^2}}\Big]\\&\quad z=\zeta_i+F_i\,,\ i=1,2,\cdots,n-1\end{aligned}\right\} \tag{5b}$$

$$\begin{aligned}&\frac{\partial\phi_1}{\partial t}-\frac{1}{2}|U_1|^2-A+\frac{1}{2}\Big[(U_1+\nabla\phi_1)^2+\Big(\frac{\partial\phi_1}{\partial z}\Big)^2\Big]+\\&g(\bar{d}_1+h)=0,\quad z=h\end{aligned} \tag{5c}$$

其中 $\nabla=\Big(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial y}\Big)$, $\tau_i(i=1,2,\cdots,n)$ 为各交界面上的表面张力系数. 在 (5a) 和 (5c) 中, 假设在远场流体静止的条件下

$$p_{n-1}(z=\zeta_{n-1}+F_{n-1})\rightarrow\rho_n g\bar{d}_n$$

$$p_1(z=\zeta_1+\bar{d}_1)\rightarrow\sum_{i=2}^{n}\rho_i g\bar{d}_i$$

$$p_1(z=h)\rightarrow\sum_{i=1}^{n}\rho_i g\bar{d}_i$$

2 Hamilton 逼近

从物理意义上说, Hamilton 泛涵 H 表示水波运动系统的总能量, 即由动能和势能构成: $H=T+V$. 对于该 n 层流体系统, 可将其表示为

$$\begin{aligned}T=&\int\mathrm{d}\boldsymbol{x}\Big\{\int_h^{\zeta_1+F_1}\mathrm{d}z\frac{1}{2}\rho_1\Big[(\nabla\Phi_1)^2+\Big(\frac{\partial\Phi_1}{\partial z}\Big)^2\Big]+\\&\int_{\zeta_1+F_1}^{\zeta_2+F_2}\mathrm{d}z\frac{1}{2}\rho_2\Big[(\nabla\Phi_2)^2+\Big(\frac{\partial\Phi_2}{\partial z}\Big)^2\Big]+\cdots+\\&\int_{\zeta_{n-1}+F_{n-1}}^{\zeta_n+F_n}\mathrm{d}z\frac{1}{2}\rho_n\Big[(\nabla\Phi_n)^2+\Big(\frac{\partial\Phi_n}{\partial z}\Big)^2\Big]\Big\}\end{aligned} \tag{6}$$

$$\begin{aligned}V=&\int\mathrm{d}\boldsymbol{x}\Big\{\int_{h+F_1}^{\zeta_1+F_1}\mathrm{d}z\int_0^z\rho_1 g\mathrm{d}z+\\&\int_{\zeta_1+F_1}^{\zeta_2+F_2}\mathrm{d}z\int_0^z\rho_2 g\mathrm{d}z+\cdots+\\&\int_{\zeta_{n-2}+F_{n-2}}^{\zeta_{n-1}+F_{n-1}}\mathrm{d}z\int_0^z\rho_{n-1}g\mathrm{d}z+\\&\int_{\zeta_{n-1}+F_{n-1}}^{\zeta_n+F_n-\bar{d}_n}\mathrm{d}z\int_0^z\rho_n g\mathrm{d}z+\\&\tau_1\Big(\sqrt{1+(\nabla\zeta_1)^2}-1\Big)+\cdots+\\&\tau_n\Big(\sqrt{1+(\nabla\zeta_n)^2}-1\Big)\Big\}=\\&\int\mathrm{d}\boldsymbol{x}\Big\{\frac{1}{2}\rho_n g(\zeta_n+F_n-\bar{d}_n)^2+\\&\frac{1}{2}g(\zeta_{n-1}+F_{n-1})^2(\rho_{n-2}-\rho_{n-1})+\cdots+\\&\frac{1}{2}g(\zeta_1+F_1)^2(\rho_1-\rho_2)-\frac{1}{2}\rho_1 g(h+\bar{d}_1)^2+\\&\tau_1\big(\sqrt{1+(\nabla\zeta_1)^2}-1\big)+\cdots+\\&\tau_n\big(\sqrt{1+(\nabla\zeta_n)^2}-1\big)\Big\}\end{aligned} \tag{7}$$

由方程 (3) 和 (4a)~(4c)，并考虑到在无穷远处速度为零的假设 [2]，可将式 (6) 化为

$$\begin{aligned}T=&\frac{1}{2}\int\mathrm{d}\boldsymbol{x}\Big\{\Big[\rho_n\Phi_n\Big(\frac{\partial\Phi_n}{\partial z}-\nabla\Phi_n\cdot\zeta_n\Big)\Big]_{z=\zeta_n+F_n}+\\&\Big[(\rho_{n-1}\Phi_{n-1}-\rho_n\Phi_n)\Big(\frac{\partial\Phi_n}{\partial z}-\nabla\Phi_n\cdot\zeta_n\Big)\Big]_{z=\zeta_{n-1}+F_{n-1}}+\\&\cdots+\Big[(\rho_1\Phi_1-\rho_2\Phi_2)\Big(\frac{\partial\Phi_1}{\partial z}-\nabla\Phi_1\cdot\nabla\zeta_1\Big)\Big]_{z=\zeta_1+F_1}-\\&\Big[\rho_1\Phi_1\Big(\frac{\partial\Phi_1}{\partial z}-\nabla\Phi_1\cdot\nabla h\Big)\Big]_{z=h}\Big\}\end{aligned} \tag{8}$$

可定义下列正则变量

$$Q_0=h,\quad R_0=-\rho_1[\Phi_1]_{z=h} \tag{9a}$$

$$\left.\begin{aligned}&Q_i=\zeta_i+F_i,\quad R_i=[\rho_i\Phi_i-\rho_{i+1}\Phi_{i+1}]_{z=\zeta_i+F_i}\\&\qquad i=1,2,\cdots,n-1\end{aligned}\right\} \tag{9b}$$

$$Q_n = \zeta_n + F_n\,, \quad R_n = \rho_n[\Phi_n]_{z=\zeta_n+F_n} \qquad (9c)$$

其中，R_0, R_i 和 R_n 可称之为动量势密度差. 至此，可给出下列 Hamilton 正则方程

$$\left.\begin{aligned} \frac{\partial Q_k}{\partial t} &= \frac{\delta H}{\delta R_k}, \\ \frac{\partial R_k}{\partial t} &= -\frac{\delta H}{\delta Q_k}, \end{aligned} \quad k = 0, 1, \cdots, n \right\} \qquad (10)$$

其中，δ 表示变分导数. 容易证明，由正则方程 (10) 可得到运动学边界条件 (4a)~(4c) 和动力学边界条件 (5a)~(5c).

3 结 论

本文着眼于分层流体运动的特征，通过建立一个在非平整海底上运动的 n 层流体系统，寻找出一组正则变量，从而得到一组 Hamilton 正则方程. 这可以看作是对单层自由表面波 Hamilton 框架的扩展和对两层流体纯波 [6] 及两层 [7], n 层 [8] 流体波 - 流相互作用 Hamilton 表示的推广.

由于经典的 Hamilton 力学是建立在守恒系统之上的，而在我们这个物质世界里几乎所有的可观察到的经典物理过程都是不守恒的. 因此，在当前十分有必要在 Hamilton 动力系统的研究中能够包含基本的耗散效应.

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HAMILTONIAN APPROXIMATION OF WATER WAVES IN A n-LAYER FLOW OVER UNEVEN BOTTOMS [1)]

Huang Hu
(*Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai* 200072, *China*)
(*LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing* 100080, *China*)

Abstract The interfacial waves on large-scale currents in the ocean is typical of stratified flow. Here, a n-layer fluid evolution system, with constant density and uniform horizontal current velocity in each layer with a thickness consisting of undisturbed and disturbed part, over uneven moving bottoms is developed to illustrate the Hamiltonian description. The rigid and moving bottoms against the free surface in the top-layer flow have a general topography. After clearly stating the governing equations for the n layers of fluids and the kinematical and dynamical boundary conditions (including the effects of the interfacial tension) holding at each interface between the fluid layers, the Hamiltonian formulation of the stratified fluid dynamical system is constructed by giving the canonical equations and the canonical variables which are the interface displacements and the difference in momentum potential density measured at the interfaces.

Key words Hamiltonian description, interfacial waves, large-scale currents, a n-layer fluid, uneven moving bottoms

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