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弹性力学混合状态方程的弱形式及其边值问题[1)]

丁克伟　　唐立民

(大连理工大学工程力学研究所 ,大连　116023)

摘要　导出了弹性力学混合状态方程和边界条件弱形式的统一方程 ,此法使函数的选择无需事先完全满足边界条件 ,对于各种不同的边值问题可以用统一形式处理 , 这使得求解弹性力学问题的形式得以扩大和统一 .

关键词　弹性力学 ,混合状态方程 ,Hamilton 正则方程 ,弱形式 ,边值问题

引　言

文[1]阐明了弹性力学混合状态方程的重要性并首次给出与之对应的从 Hellinger - Reissner 变分以及修正后导出的 Hamilton 正则方程 ,同时也指出求解混合状态方程组是一个有广泛应用但现在还没有很好开拓的领域 ,这是因为在连续介质中的 Hamilton 方程还要满足复杂的边界条件 ,这也就是它的难点 . 文[2～4]曾使用这组方程解决了叠层板壳问题 ,并且克服了文[5～7]需处理随层数增加而增加的大量未知量问题 . 但这些解法由于都采用了强形式的平衡方程和边界条件 ,使得在求解时面对不同的边值问题必须采用不同的处理方法 ,且过多的依赖于技巧性 ,因此难以得到推广 . 特别对于在复杂混合边界条件 ,无法得到解析解的 . 本文针对这个问题 ,在文[1][2]的启发下 ,导出了弹性力学混合状态方程和边界条件弱形式的统一方程 ,此法使函数的选择无需事先完全满足边界条件 ,对于各种不同的边值问题可以用统一形式处理 ,这使得求解该类问题的形式得以扩大和统一 .

1　Hellinger - Reissner 变分原理和 Hamilton 正则方程

依据三维弹性力学问题 Hellinger - Reissner 变分原理 ,可以写出具有任意边界条件以 σ_x , σ_y , σ_z , τ_{yz} , τ_{zx} , τ_{xy} , u , v , w ,为独立变量的泛函如下 :

$$\delta U = \delta \iiint_V \left\{ \sigma_z \frac{\partial w}{\partial z} + \tau_{yz} \frac{\partial v}{\partial z} + \tau_{zx} \frac{\partial u}{\partial z} - H \right\} dV + \Gamma \tag{1}$$

其中

$$-H = \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \tau_{yz} \frac{\partial w}{\partial y} + \tau_{zx} \frac{\partial w}{\partial x} - B(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}) \tag{2}$$

是 Hamilton 函数的一个二次式 . 以上符号都是常规的 . 对于各向异性材料

$$B(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}) =$$

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$$\frac{1}{2}[\sigma_x,\sigma_y,\sigma_z,\tau_{yz},\tau_{zx},\tau_{xy}]\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & sym & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}\begin{bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy}\end{bmatrix} \tag{3}$$

续 是因 介质 采 解法 用了

S_{ij}为本构关系的柔度系数

$$\Gamma = \iint_s [\lambda_x(p_x - \bar{p}_x)u + \lambda_y(p_y - \bar{p}_y)v + \lambda_z(p_z - \bar{p}_z)w + [(1-\lambda_x)\bar{u} - u]p_x + [(1-\lambda_y)\bar{v} - v]p_y + [(1-\lambda_z)\bar{w} - w]p_z]\mathrm{d}s \tag{4}$$

(4)式中的 $\lambda_x,\lambda_y,\lambda_z$定义在边界上的每一点,它们的取值规则为:

$s \in S_\sigma$(应力边界), $\lambda_x = \lambda_y = \lambda_z = 1$,

$s \in S_u$(位移边界), $\lambda_x = \lambda_y = \lambda_z = 0$

$s \in S_{\sigma u}$(应力 - 位移边界),在某一方向上(如 x)为力的边界条件,则 $\lambda_x = 1$,否则为零.

(4)式代表了最一般的边界条件,包括混合边界条件.

对(1)式变分运算,得

$$\left.\begin{aligned} &\iiint_V \left(\frac{\partial \tau_{zx}}{\partial z} + \frac{\partial H}{\partial u}\right)\delta u \,\mathrm{d}V + \iint_{s_\sigma}(\bar{p}_x - p_x)\delta u \,\mathrm{d}s = 0 \\ &\iiint_V \left(\frac{\partial \tau_{yz}}{\partial z} + \frac{\partial H}{\partial v}\right)\delta v \,\mathrm{d}V + \iint_{s_\sigma}(\bar{p}_y - p_y)\delta v \,\mathrm{d}s = 0 \\ &\iiint_V \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial H}{\partial w}\right)\delta w \,\mathrm{d}V + \iint_{s_\sigma}(\bar{p}_z - p_z)\delta w \,\mathrm{d}s = 0 \end{aligned}\right\}$$

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$$\tag{5b}$$

$$\left.\begin{aligned} &\iiint_V \left(\frac{\partial u}{\partial x} - \frac{\partial H}{\partial \sigma_x}\right)\delta \sigma_x \,\mathrm{d}V + \iint_{s_u}(\bar{u} - u)n_x \delta\sigma_x \,\mathrm{d}s = 0 \\ &\iiint_V \left(\frac{\partial v}{\partial y} - \frac{\partial H}{\partial \sigma_y}\right)\delta \sigma_y \,\mathrm{d}V + \iint_{s_u}(\bar{v} - v)n_y \delta\sigma_y \,\mathrm{d}s = 0 \\ &\iiint_V \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial H}{\partial \tau_{xy}}\right)\delta \tau_{xy} \,\mathrm{d}V + \iint_{s_u}[(\bar{v} - v)n_x + (\bar{u} - u)n_y]\delta\tau_{xy} \,\mathrm{d}s = 0 \end{aligned}\right\}$$

动力 解及

采用记号:

$$\mathrm{q} = (u, v, w)^T, \mathrm{p} = (\tau_{zx}, \tau_{yz}, \sigma_z)^T, \mathrm{p}_1 = (\sigma_x, \sigma_y, \tau_{xy})^T \tag{6}$$

则(5)式可以写成以下形式:

$$\left.\begin{aligned} &\iiint_V \left(\frac{\partial \mathrm{p}}{\partial z} + \frac{\partial H}{\partial \mathrm{q}}\right) \delta \mathrm{q} \, \mathrm{d}V + \Gamma_1 = 0 \\ &\iiint_V \left(\frac{\partial \mathrm{q}}{\partial z} - \frac{\partial H}{\partial \mathrm{p}}\right) \delta \mathrm{p} \, \mathrm{d}V + \Gamma_2 = 0 \\ &\iiint_V \left(D\mathrm{q} - \frac{\partial H}{\partial \mathrm{p}_1}\right) \delta \mathrm{p}_1 \, \mathrm{d}V + \Gamma_3 = 0 \end{aligned}\right\}$$

∂

$$\delta_{p_1} = \begin{bmatrix} \delta\sigma_x & 0 & 0 \\ 0 & \delta\sigma_y & 0 \\ 0 & 0 & \delta\tau_{xy} \end{bmatrix}, \quad D_2 = \begin{bmatrix} S_{15} & S_{14} & S_{13} & -\frac{\partial}{\partial x} & 0 & 0 \\ S_{25} & S_{24} & S_{23} & 0 & -\frac{\partial}{\partial y} & 0 \\ S_{65} & S_{64} & S_{63} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{21} & S_{22} & S_{26} \\ S_{61} & S_{62} & S_{66} \end{bmatrix}, \qquad S_2 = \begin{bmatrix} (u - \bar{u}) n_x \\ (v - \bar{v}) n_y \\ (u - \bar{u}) n_y + (v - \bar{v}) n_x \end{bmatrix}$$

为了求解(9)和(10)式,设

$$F = F(z) N_F(x,y), \quad p_1 = p(z) N_P(x,y) \tag{11}$$

将(11)分别先后代入(10)、(9)式,在 x,y 面内完成二重积分后,由(10)解出 $p(z)$ 再代入(9)式经过化简,可得

$$\frac{d}{dz} F(z) = A F(z) + S(z) \tag{12}$$

式(12)的解为:

$$F(z) = e^{Az} F(0) + \int_0 e^{A(z-\tau)} S(\tau) d\tau \tag{13}$$

3　应用和算例

例 1　考察一均布压力 q 作用下三层两对边固支$(x = 0, a)$,两对边简支板$(y = 0, b)$,几何参数为:$a = b$,$h_1 = h_3 = 0.1h$,$h_2 = 0.8h$. 第一和第三层材料相同,材料特性为:

$C_{22}/C_{11} = 0.543103$,$C_{12}/C11 = 0.246269$

$C_{33}/C_{11} = 0.530172$,$C_{44}/C11 = 0.266810$

$C_{23}/C_{11} = 0.115017$,$C_{13}/C11 = 0.083172$

$C_{55}/C_{11} = 0.159914$,$C_{66}/C11 = 0.262931$

令 ρ 为第一层材料与第二层材料 C_{11} 比值,若 $\rho = 1$ 则为单层板,若 $\rho \neq 1$ 则为三层板.

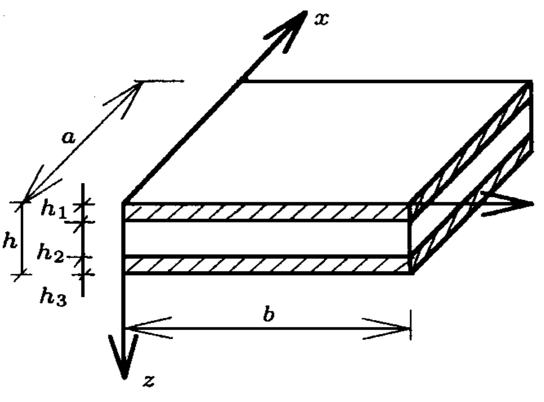

图 1　三层板

Fig. 1　Three laminated plate

令

$$u = u_x + \left(1 - \frac{x}{a}\right) \bar{u}^{(0)}_{(y,z)} + \frac{x}{a} \bar{u}^{(a)}_{(y,z)} \tag{14}$$

其中 $\bar{u}^{(0)}_{(y,z)}$ 和 $\bar{u}^{(a)}_{(y,z)}$ 分别是$(x = 0, a)$处的位移函数,目的是为了使 u 在两端为零.

取(11)式中的状态向量函数如下:

$$u_x = \sum_m \sum_n u_{x,mn}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad \tau_{zx} = \sum_m \sum_n \tau_{zx,mn}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$v = \sum_m \sum_n v_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \qquad \tau_{yz} = \sum_m \sum_n \tau_{yz,mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w = \sum_m \sum_n w_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \sigma_z = \sum_m \sum_n \sigma_{z,mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

将(14) 式和上面的向量函数代入(10) - (9)式,然后利用状态向量 F 在层间的连续性及状态转移矩阵(见文[2～4]),问题即可求解. 此时尚未满足的边界条件可以由在 $x = 0, a$ 处的 $\iint_x (u - \bar{u}) n_x \mathrm{d}s = 0$ 得到满足, 边界未知的位移函数 $\bar{u}^{(0)}_{(y,z)}$ 和 $\bar{u}^{(a)}_{(y,z)}$ 也是因此而得到的(限于篇幅,详细过程从略). 计算结果见表 1.

表 1 对边固支的单层板和三层板的挠度和应力

Table 1 Deflection and stresses for single - ply and three - plied laminated plate with two clamped

		$\rho=1(h/a=0.1)$			$\rho=5(h/a=0.1)$			$\rho=10(h/a=0.1)$		
		$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q	$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q	$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q
present	1 +	339.36	- 22.71	- 11.26	153.85	- 38.66	- 22.12	108.76	- 42.70	- 28.06
	1 -	339.69	- 17.63	- 8.890	154.00	- 28.37	- 16.92	108.86	- 29.16	- 20.88
	2 +	339.69	- 17.63	- 8.890	154.00	- 5.796	- 3.552	108.86	- 3.051	- 2.276
	2 -	338.91	17.59	8.827	153.23	5.655	3.424	108.09	2.907	2.117
	3 +	338.91	17.59	8.827	153.23	28.30	17.16	108.09	29.14	21.27
	3 -	338.38	22.55	11.17	153.03	38.53	22.28	107.97	42.56	28.32
SAP5	1 +	335.56	- 24.84	- 10.88	154.86	- 44.48	- 22.75	110.47	- 51.13	- 29.69
	1 -	336.68	- 18.49	- 8.136	155.20	- 28.26	- 15.95	110.65	- 26.85	- 19.60
	2 +	336.68	- 18.49	- 8.136	155.20	- 6.197	- 3.943	110.65	- 3.224	- 2.706
	2 -	336.36	18.57	8.385	154.60	6.098	3.853	109.91	3.104	2.547
	3 +	336.36	18.57	8.385	154.60	28.40	16.38	109.91	27.18	20.11
	3 -	335.32	24.89	11.16	154.26	44.33	23.24	109.72	50.70	30.24

w, σ_x, σ_y on $x = a/2, y = b/2$.

1 + : Upper surface of top ply, 1 - : Lower surface of top ply; 2 : Mid ply, 3 : Bottom ply

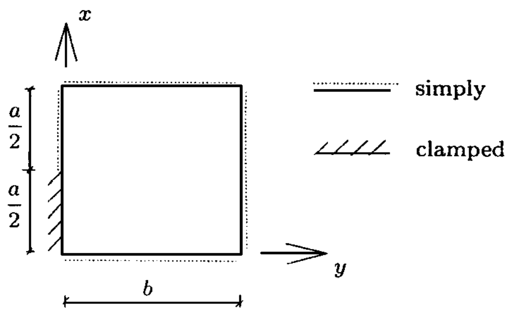

图 2 混合边界板

Fig. 2 Mixed boundary plate

例 2 考虑一均布压力 q 作用下具有混合边界条件的三层板(图 2). 仍选用例 1 的试函数. 几何参数和材料特性同例 1. 该问题尚未满足的边界条件可以由

$$\int_0^a \int (v - \bar{v}) n_y \mathrm{d}x\mathrm{d}z = 0$$

得到满足. 计算结果见表 2.

表 2　具有混合边界条件的单层板和三层板的挠度和应力

Table 2　Deflection and stresses for single - ply and three - plied laminated plate with mixed boundary

		$\rho=5(h/a=0.4)$			$\rho=5(h/a=0.8)$			$\rho=1(h/a=0.2)$		
		$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q	$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q	$wc_{11}^{(2)}/qh$	σ_x/q	σ_y/q
present	1 +	3.7000	- 3.778	- 3.407	1.1497	- 2.1265	- 1.8488	47.707	- 9.0523	- 5.2683
	1 -	3.6583	- 1.678	- 2.031	1.1201	0.1110	- 0.4413	47.745	- 7.1306	- 4.0754
	2 +	3.6583	- 0.456	- 0.573	1.1201	- 0.1011	- 0.2587	47.745	- 7.1306	- 4.0754
	2 -	2.8658	0.131	0.441	0.2961	- 0.0728	0.0640	46.958	5.3591	4.3439
	3 +	2.8658	0.689	2.251	0.2961	- 0.3266	0.3717	46.958	5.3591	4.3439
	3 -	2.8398	2.864	3.664	0.2899	0.6168	0.9724	46.722	7.3445	5.5305
SAP5	1 +	3.6385	- 3.950	- 3.591	1.0829	- 2.1723	- 1.7509	44.418	- 9.2218	- 5.2960
	1 -	3.6511	- 1.028	- 1.797	1.0660	0.2629	- 0.3314	44.713	- 6.3007	- 4.1775
	2 +	3.6511	- 0.328	- 0.540	1.0660	- 0.0182	- 0.2036	44.713	- 6.3007	- 4.1775
	2 -	2.8313	0.211	0.406	0.2601	- 0.0726	0.0244	43.925	6.1115	3.9556
	3 +	2.8313	1.213	2.226	0.2601	- 0.1814	0.4047	43.925	6.1115	3.9556
	3 -	2.8054	3.405	3.747	0.2528	0.7523	1.0226	43.668	7.4405	4.9881

w, σ_x, σ_y on $x = a/2, y = b/2$.

1 + : Upper surface of top ply, 1 - : Lower surface of top ply; 2: Mid ply, 3: Bottom ply

4　结　论

本文给出的弹性力学弱形式求解混合状态方程是有效的,物理概念清晰,边值问题处理统一,无需特殊技巧,使得求解问题的形式得以扩大,这在力学计算上或许是一大进步. 本方法完全可以用于板壳结构的动力计算问题.

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WEAK FORMULATION OF MIXED STATE EQUATION AND BOUNDARY VALUE PROBLEM OF THEORY OF ELASTICITY[1)]

Ding Kewei Tang Limin

(Research Institute of Engineering Mechanics, Dalian University of Technology, Dalian 116023*, China)*

Abstract Prof. Tang clarified importance of mixed state equation of elasticity, and first gave Hamilton canonical equation by modifying Hellinger - Reissner variational principle. At the same time author pointed out that the study of solution of mixed state equation has still not been well developed though it has a wide spread application prospect. This is because Hamilton equation must satisfy comples boundary condition in continuous medium mechanics. which is its difficult point, Fan and Ding applied the mixed state equation to investigated laminated plate and shell, and overcame the disadvantage in the ordinary successive approximations method, in which a lot of unknowns that appear at the real or fictitious interfaces had to be dealt with. But all of above papers adopted rigorous equilibrium and boundary conditions, their solution can be only relied on special technique. So those methods would be difficult to be popularized. In addition, it is impossible to obtained the analytical solution for plates with complex boundary conditions. In this paper, by introducing Hellinger - Reissner variational principle, universal weak formulation of mixed state equation including boundary condition are presented. An unified approach and the general solution of these equations are also given. Plates with different boundary condition can be dealt with by uniform formulas. The present study make trial functions which need not satisfied all the boundary conditions in prior, and extends and unifies the solution of the theory of elasticity.

Key words theory of elasticity, mixed state equation, hamilton canonical equation, weak formulation, boundary value problem

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