一类多自由度强非线性振动系统主共振的渐近解法

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摘要　提出一种渐近方法用来处理一类多自由度强非线性自治振动系统，它是新渐近方法 [1] 的推广. 本方法适用于主共振情形，我们建立了振幅和相位所满足的方程. 文末用两个例子说明本方法的有效性.

关键词　多自由度强非线性系统，渐近方法，周期解，主共振

引　言

大家知道，经典的摄动方法仅适用于弱非线性系统. 因此，寻求可应用于强非线性系统的分析方法就成为一个重要研究课题. 近十多年来，对于单自由度强非线性振动系统

$$\ddot{u}+g(u)=\varepsilon f(u,\dot{u}) \tag{1}$$

已提出多种摄动方法，主要有渐近方法 [1,2], 时间变换法 [3,4], 广义平均法 [5], 非线性尺度法 [6], 推广的 KB 法 [7] 和频闪法 [8] 等. 相比之下，人们对多自由度非线性振动系统的研究远不如对单自由度系统来得深入 [9], 目前还局限于弱非线性系统的研究 [10,11], 而强非线性情形尚未研究.

本文研究下列多自由度强非线性自治系统

$$\ddot{u}_i+g_i(u_i)=\varepsilon f_i(u_1,\cdots,u_N,\dot{u}_1,\cdots,\dot{u}_N)\qquad (i=1,2,\cdots,N) \tag{2}$$

其中 g_i 和 f_i 都是非线性函数， ε 为小参数， $0<\varepsilon<<1$, 我们考虑主共振情形，推广和改进新的渐近方法 [1], 给出系统 (2) 解的振幅和相位所应满足的方程，从而求出周期解. 作为应用，研究了两个例子，说明本方法的有效性.

1　未扰动解

我们先研究未扰动系统 ($\varepsilon=0$)

$$\ddot{u}_i+g_i(u_i)=0\qquad (i=1,2,\cdots,N) \tag{3}$$

1994-01-19 收到第一稿， 1994-04-28 收到修改稿.

设方程 (3) 有周期解. 作时间变换

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t}=\Phi_{i0}(\varphi_i) \tag{4}$$

其中 $\Phi_{i0}(\varphi_i)$ 是 φ_i 的以 2π 为周期的函数，使在新时间 φ_i 领域内方程 (3) 的解可表示为

$$u_i=a_i\cos\varphi_i+b_i \tag{5}$$

其中 a_i 为振幅， b_i 为偏心值，而方程 (3) 变为

$$\Phi_{i0}\frac{\mathrm{d}}{\mathrm{d}\varphi_i}(\Phi_{i0}u_i')+g_i(u_i)=0 \tag{6}$$

式中 $u_i'=\mathrm{d}u_i/\mathrm{d}\varphi_i=-a_i\sin\varphi_i$, 方程 (6) 两边乘以 u_i' 然后对 φ_i 由 0 开始积分得

$$\frac{1}{2}(a_i\Phi_{i0}\sin\varphi_i)^2=v_i(a_i+b_i)-v_i(a_i\cos\varphi_i+b_i) \tag{7}$$

式中

$$v_i(u_i)=\int_0^{u_i}g_i(x)\mathrm{d}x \tag{8}$$

由方程 (7) 求得时间变换函数 Φ_{i0} 如下

$$\Phi_{i0}(\varphi_i)=\sqrt{\frac{2[v_i(a_i+b_i)-v_i(a_i\cos\varphi_i+b_i)]}{a_i^2\sin^2\varphi_i}}\overset{\text{def}}{=}\Phi_{i0}(a_i,\varphi_i) \tag{9}$$

在方程 (7) 中令 $\varphi_i=\pi$, 得

$$v_i(a_i+b_i)-v_i(-a_i+b_i)=0 \tag{10}$$

因此， b_i 可以看作是 a_i 的函数.

由于 $\Phi_{i0}^{-1}(a_i,\varphi_i)$ 是 φ_i 的 2π 周期函数，故可用 Fourier 级数表示，令

$$\Phi_{i0}^{-1}(a_i,\varphi_i)=c_{i0}(a_i)+\sum_{n=1}^{M}c_{in}(a_i)\cos n\varphi_i \tag{11}$$

则由式 (4) 可得周期解的周期 $T_{i0}(a_i)$ 为

$$T_{i0}(a_i)=\int_0^{2\pi}\Phi_{i0}^{-1}(a_i,\varphi_i)\mathrm{d}\varphi_i=2\pi c_{i0}(a_i) \tag{12}$$

2 渐近解法

现在考虑 $\varepsilon\neq 0$ 的情形，我们设方程组 (2) 的解为如下形式

$$u_i=a_i\cos\varphi_i+b_i+\varepsilon x_{i1}(a_1,\cdots,a_N)+O(\varepsilon^2) \tag{13}$$

$$\frac{\mathrm{d}a_i}{\mathrm{d}t}=\varepsilon A_{i1}(a_1,\cdots,a_N)+O(\varepsilon^2) \tag{14}$$

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t}=\Phi_{i0}(a_i,\varphi_i)+\varepsilon\Phi_{i1}(a_1,\cdots,a_N,\varphi_1,\cdots,\varphi_N)+O(\varepsilon^2) \tag{15}$$

其中 $\Phi_{i0}(a_i,\varphi_i)$ 由式 (9) 确定， a_i 和 b_i 满足式 (10). 由式 (15) 消去 dt 得

$$\frac{\mathrm{d}\varphi_1}{\Phi_{10}+\varepsilon\Phi_{11}}=\frac{\mathrm{d}\varphi_2}{\Phi_{20}+\varepsilon\Phi_{21}}=\cdots=\frac{\mathrm{d}\varphi_N}{\Phi_{N0}+\varepsilon\Phi_{N1}} \tag{16}$$

对于周期解的周期 T_i 为

$$T_i=\int_0^{2\pi}\frac{\mathrm{d}\varphi_i}{\Phi_{i0}+\varepsilon\Phi_{i1}}=\int_0^{2\pi}\Phi_{i0}^{-1}\mathrm{d}\varphi_i+O(\varepsilon)=2\pi c_{i0}(a_i)+O(\varepsilon) \tag{17}$$

我们将考虑主共振情形 (内共振)

$$T_1=T_2=\cdots=T_N \tag{18}$$

由式 (17) 得

$$c_{10}(a_1)=c_{20}(a_2)+O(\varepsilon)=\cdots=c_{N0}(a_N)+O(\varepsilon) \tag{19}$$

现在把式 (11) 代入式 (16), 积分得

$$\begin{aligned} & c_{10}\varphi_1+\sum_{n=1}^{M}c_{1n}\sin n\varphi_1/n \\ = & c_{20}\varphi_2+\sum_{n=1}^{M}c_{2n}\sin n\varphi_2/n+\bar{\gamma}_2+O(\varepsilon) \\ = & \cdots\cdots \\ = & c_{N0}\varphi_N+\sum_{n=1}^{M}c_{Nn}\sin n\varphi_N/n+\bar{\gamma}_N+O(\varepsilon) \end{aligned} \tag{20}$$

其中 $\bar{\gamma}_2,\cdots,\bar{\gamma}_N$ 为积分常数. 为方便起见我们补充定义 $\bar{\gamma}_1=0$. 因此，利用式 (19) 可把式 (20) 改写为

$$\varphi_i=\varphi_j+\sum_{n=1}^{M}\frac{1}{nc_{i0}}(c_{jn}\sin n\varphi_j-c_{in}\sin n\varphi_i)+c_{i0}^{-1}(\bar{\gamma}_j-\bar{\gamma}_i)+O(\varepsilon) \tag{21}$$

$i\neq j$. 上式建立了 φ_i 与 φ_j 的函数关系，进一步我们令

$$\varphi_i=\varphi_j+\gamma_j-\gamma_i+L_{i0}+\sum_{k=1}^{M}(L_{ik}\cos k\varphi_j+M_{ik}\sin k\varphi_j)+O(\varepsilon)\overset{\mathrm{def}}{=}\Psi_{ij}(\varphi_j) \tag{22}$$

其中 $\gamma_j=c_{j0}^{-1}\bar{\gamma}_j(j=1,\cdots,N)$, L_{i0}, L_{ik} 和 M_{ik} 为待定 Fourier 系数，现在把式 (22) 代入式 (21) 得

$$\begin{aligned} & L_{i0}+\sum_{k=1}^{M}(L_{ik}\cos k\varphi_j+M_{ik}\sin k\varphi_j) \\ & =\sum_{n=1}^{M}\frac{1}{nc_{i0}}\left[c_{jn}\sin n\varphi_j-c_{in}\sin n\Psi_{ij}(\varphi_j)\right]+O(\varepsilon) \end{aligned} \tag{23}$$

再应用谐波平衡法得

$$\left.\begin{aligned} L_{i0} &= \frac{1}{2\pi}\int_0^{2\pi} G_i(a,\gamma,L,M,\varphi_j)\mathrm{d}\varphi_j \\ L_{ik} &= \frac{1}{\pi}\int_0^{2\pi} G_i(a,\gamma,L,M,\varphi_j)\cos k\varphi_j \mathrm{d}\varphi_j \\ M_{ik} &= \frac{1}{\pi}\int_0^{2\pi} G_i(a,\gamma,L,M,\varphi_j)\sin k\varphi_j \mathrm{d}\varphi_j \end{aligned}\right\} \tag{24}$$

其中 $k=1,2,\cdots,M$, $a=(a_1,\cdots,a_N)$, $\gamma=(\gamma_2,\cdots,\gamma_N)$, $L=(L_{i0},L_{i1},\cdots,L_{iN})$, $M=(M_{i1},\cdots,M_{iN})$.

$$G_i(a,\gamma,L,M,\varphi_j)=\sum_{n=1}^{M}\frac{1}{nc_{i0}}\left[c_{jn}\sin n\varphi_j - c_{in}\sin n\Psi_{ij}(\varphi_j)\right] \tag{25}$$

至此我们确定了 φ_i 与 φ_j 的关系式 (22). 为方便起见再补充定义 $\Psi_{ii}(\varphi_i)\equiv\varphi_i$.

下面确定式 (13)—(15) 中的 x_{i1}, A_{i1} 和 Φ_{i1}. 对 u_i 求导得

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -a_i\Phi_{i0}\sin\varphi_i + \varepsilon(A_{i1}h_i + A_{i1}\cos\varphi_i - a_i\Phi_{i1}\sin\varphi_i) + O(\varepsilon^2) \tag{26}$$

$$\begin{aligned} \frac{\mathrm{d}^2u_i}{\mathrm{d}t^2} = &-a_i\Phi_{i0}\frac{\partial\Phi_{i0}}{\partial\varphi_i}\sin\varphi_i - a_i\Phi_{i0}^2\cos\varphi_i \\ &-\varepsilon\Big[2a_i\Phi_{i0}\Phi_{i1}\cos\varphi_i + \Big(2A_{i1}\Phi_{i0} + a_iA_{i1}\frac{\partial\Phi_{i0}}{\partial a_i} \\ &+a_i\Phi_{10}\frac{\partial\Phi_{i1}}{\partial\varphi_i} + a_i\Phi_{i1}\frac{\partial\Phi_{i0}}{\partial\varphi_i}\Big)\sin\varphi_i\Big] + O(\varepsilon^2) \end{aligned} \tag{27}$$

其中

$$h_i = \frac{\mathrm{d}b_i}{\mathrm{d}a_i} = [g_i(-a_i+b_i)+g_i(a_i+b_i)]/[g_i(-a_i+b_i)-g_i(a_i+b_i)] \tag{28}$$

再把 $g_i(u_i)$ 展开

$$g_i(u_i) = g_i(a_i\cos\varphi_i+b_i) + \varepsilon x_{i1}g_i'(a_i\cos\varphi_i+b_i) + O(\varepsilon^2) \tag{29}$$

式中撇号表示对 u_i 求导.

把式 (27) 和 (29) 代入方程组 (2), 令等式两边 ε 的一次幂系数相等，得

$$\begin{aligned} a_i\frac{\partial}{\partial\varphi_i}(\Phi_{i0}\Phi_{i1}\sin^2\varphi_i) = &-f_{i0}(a_1,\cdots,a_N,\varphi_i)\sin\varphi_i \\ &-A_{i1}\left(2\Phi_{i0}+a_i\frac{\partial\Phi_{i0}}{\partial a_i}\right)\sin^2\varphi_i + x_{i1}g_i'(a_i\cos\varphi_i+b_i)\sin\varphi_i \end{aligned} \tag{30}$$

其中

$$\begin{aligned} f_{i0}(a_1,\cdots,a_N,\varphi_i) = f_i(&a_1\cos\varphi_1+b_1,\cdots,a_N\cos\varphi_N+b_N, \\ &-a_1\Phi_{10}\sin\varphi_1,\cdots,-a_N\Phi_{N0}\sin\varphi_N) \end{aligned} \tag{31}$$

$$\varphi_j = \Psi_{ji}(\varphi_i)\,, \qquad j=1,2,\cdots,N$$

方程 (30) 对 φ_i 积分得

$$\begin{aligned} a_i\Phi_{i0}\Phi_{i1}\sin^2\varphi_i = & -\int_0^{\varphi_i} f_{i0}(a_1,\cdots,a_N,\varphi_i)\sin\varphi_i \mathrm{d}\varphi_i \\ & -A_{i1}\int_0^{\varphi_i}\left(2\Phi_{i0}+a_i\frac{\partial\Phi_{i0}}{\partial a_i}\right)\sin^2\varphi_i\mathrm{d}\varphi_i \\ & +x_{i1}[g_i(a_i+b_i)-g_i(a_i\cos\varphi_i+b_i)]/a_i \end{aligned} \tag{32}$$

在上式中分别令 $\varphi_i=\pi,\ 2\pi$, 我们得到

$$\begin{aligned} x_{i1} = & \ a_i\Big[\int_0^{\pi} f_{i0}(a_1,\cdots,a_N,\varphi_i)\sin\varphi_i\mathrm{d}\varphi_i \\ & +A_{i1}\int_0^{\pi}\left(2\Phi_{i0}+a_i\frac{\partial\Phi_{i0}}{\partial a_i}\right)\sin^2\varphi_i\mathrm{d}\varphi_i\Big]\Big/ \\ & [g_i(a_i+b_i)-g_i(-a_i+b_i)] \end{aligned} \tag{33}$$

$$A_{i1}=-\int_0^{2\pi} f_{i0}(a_1,\cdots,a_N,\varphi_i)\sin\varphi_i\mathrm{d}\varphi_i\Big/\int_0^{2\pi}\left(2\Phi_{i0}+a_i\frac{\partial\Phi_{i0}}{\partial a_i}\right)\sin^2\varphi_i\mathrm{d}\varphi_i \tag{34}$$

于是 Φ_{i1} 由式 (32) 完全确定，若有必要可继续求高阶近似解.

周期解对应于 $A_{i1}(a_1,\cdots,a_N)=0$ 和式 (18), 共 $2N-1$ 个方程， $2N-1$ 个未知量 $a_1,\cdots,a_N$ 和 $\gamma_2,\cdots,\gamma_N$.

3 耦合广义 Van der Pol 振子

作为本方法的应用，我们研究两类耦合广义 Van Der Pol 振子.

例 1 先考虑有对称恢复力的情形

$$\left.\begin{aligned} \ddot{u}_1+m_1u_1+m_3u_1^3=\varepsilon(1-u_1^2)\dot{u}_1+\varepsilon\lambda_1u_2 \\ \ddot{u}_2+k_1u_2+k_3u_2^3=\varepsilon(1-u_2^2)\dot{u}_2+\varepsilon\lambda_2u_1 \end{aligned}\right\} \tag{35}$$

这里 $f_1(u_1,u_2,\dot{u}_1,\dot{u}_2)=(1-u_1^2)\dot{u}_1+\lambda_1u_2$, $f_2(u_1,u_2,\dot{u}_1,\dot{u}_2)=(1-u_2^2)\dot{u}_2+\lambda_2u_1$, $g_1(u_1)=m_1u_1+m_3u_1^3$, $g_2(u_2)=k_1u_2+k_3u_2^3$. 因此， $v_1(u_1)=\frac{1}{2}m_1u_1^2+\frac{1}{4}m_3u_1^4$, $v_2(u_2)=\frac{1}{2}k_1u_2^2+\frac{1}{4}k_3u_2^4$. 由式 (9) 和 (10) 得

$$b_1=0\ ,\qquad \Phi_{10}(a_1,\varphi_1)=\left\{m_1+0.75m_3a_1^2+0.25m_3a_1^2\cos2\varphi_1\right\}^{1/2} \tag{36}$$

$$b_2=0\ ,\qquad \Phi_{20}(a_2,\varphi_2)=\left\{k_1+0.75k_3a_2^2+0.25k_3a_2^2\cos2\varphi_2\right\}^{1/2} \tag{37}$$

再由式 (11) 求 $\Phi_{i0}^{-1}(a_i, \varphi_i)$ 的 Fourier 级数

$$\left.\begin{aligned} c_{i0} &= \frac{1}{2\pi}\int_0^{2\pi} \Phi_{i0}^{-1}(a_i,\varphi_i)\mathrm{d}\varphi_i \\ c_{i\,2n} &= \frac{1}{\pi}\int_0^{2\pi} \Phi_{i0}^{-1}(a_i,\varphi_i)\cos 2n\varphi_i \mathrm{d}\varphi_i \\ c_{i\,2n+1} &= 0 \end{aligned}\right\} \tag{38}$$

式 (21) 变为

$$\varphi_1 = \varphi_2 + \sum_{n=1}^{M} \frac{1}{2nc_{10}}(c_{2\,2n}\sin 2n\varphi_2 - c_{1\,2n}\sin 2n\varphi_1) + \gamma_2 + O(\varepsilon) \tag{39}$$

$$\varphi_2 = \varphi_1 + \sum_{n=1}^{M} \frac{1}{2nc_{20}}(c_{1\,2n}\sin 2n\varphi_1 - c_{2\,2n}\sin 2n\varphi_2) - \gamma_2 + O(\varepsilon) \tag{40}$$

φ_1 与 φ_2 的函数形式由式 (22)—(25) 表示，其中

$$G_i(a,\gamma,L,M,\varphi_j) = \sum_{n=1}^{M} \frac{1}{2nc_{i0}}[c_{j\,2n}\sin 2n\varphi_j - c_{i\,2n}\sin 2n\Psi_{ij}(\varphi_j)] \tag{41}$$

把以上结果代入式 (32)—(34), 得

$$x_{11} = 0\,, \qquad x_{21} = 0 \tag{42}$$

$$A_{11} = \frac{\displaystyle\int_0^{2\pi}[(1-a_1^2\cos^2\varphi_1)a_1\Phi_{10}\sin^2\varphi_1 - \lambda_1 a_2\cos\varphi_2\sin\varphi_1]\mathrm{d}\varphi_1}{\displaystyle\int_0^{2\pi}\left(2\Phi_{10} + a_1\frac{\partial\Phi_{10}}{\partial a_1}\right)\sin^2\varphi_1\mathrm{d}\varphi_1} \tag{43}$$

$$A_{21} = \frac{\displaystyle\int_0^{2\pi}[(1-a_2^2\cos^2\varphi_2)a_2\Phi_{20}\sin^2\varphi_2 - \lambda_2 a_1\cos\varphi_1\sin\varphi_2]\mathrm{d}\varphi_2}{\displaystyle\int_0^{2\pi}\left(2\Phi_{20} + a_2\frac{\partial\Phi_{20}}{\partial a_2}\right)\sin^2\varphi_2\mathrm{d}\varphi_2} \tag{44}$$

$$\begin{aligned}\Phi_{11} = \Big\{ &\int_0^{\varphi_1}[(1-a_1^2\cos^2\varphi_1)a_1\Phi_{10}\sin^2\varphi_1 - \lambda_1 a_2\cos\varphi_2\sin\varphi_1]\mathrm{d}\varphi_1 \\ &-A_{11}\int_0^{\varphi_1}\left(2\Phi_{10} + a_1\frac{\partial\Phi_{10}}{\partial a_1}\right)\sin^2\varphi_1\mathrm{d}\varphi_1\Big\}\Big/a_1\Phi_{10}\sin^2\varphi_1\end{aligned} \tag{45}$$

$$\begin{aligned}\Phi_{21} = \Big\{ &\int_0^{\varphi_2}[(1-a_2^2\cos^2\varphi_2)a_2\Phi_{20}\sin^2\varphi_2 - \lambda_2 a_1\cos\varphi_1\sin\varphi_2]\mathrm{d}\varphi_2 \\ &-A_{21}\int_0^{\varphi_2}\left(2\Phi_{20} + a_2\frac{\partial\Phi_{20}}{\partial a_2}\right)\sin^2\varphi_2\mathrm{d}\varphi_2\Big\}\Big/a_2\Phi_{20}\sin^2\varphi_2\end{aligned} \tag{46}$$

为了求周期解我们令 $A_{11}=0$, $A_{21}=0$ 和

$$\int_0^{2\pi}[\Phi_{10}+\varepsilon\Phi_{11}]^{-1}\mathrm{d}\varphi_1=\int_0^{2\pi}[\Phi_{20}+\varepsilon\Phi_{21}]^{-1}\mathrm{d}\varphi_2 \tag{47}$$

得

$$\cos\gamma_2=\int_0^{2\pi}\left[(1-a_1^2\cos^2\varphi_1)a_1\Phi_{10}\sin^2\varphi_1\mathrm{d}\varphi_1/\lambda_1 a_2\int_0^{2\pi}\cos\Psi_1^*\sin\varphi_1\mathrm{d}\varphi_1\right.$$
$$-\sin\gamma_2\int_0^{2\pi}\sin\Psi_1^*\sin\varphi_1\mathrm{d}\varphi_1\Big/\int_0^{2\pi}\cos\Psi_1^*\sin\varphi_1\mathrm{d}\varphi_1 \tag{48}$$

$$a_2=\left[\int_0^{2\pi}a_2^3\Phi_{20}\cos^2\varphi_2\sin^2\varphi_2\mathrm{d}\varphi_2+\lambda_2 a_1\int_0^{2\pi}\cos\varphi_1\sin\varphi_2\mathrm{d}\varphi_2\right]\Big/$$
$$\int_0^{2\pi}\Phi_{20}\sin^2\varphi_2\mathrm{d}\varphi_2 \tag{49}$$

$$a_1=\int_0^{2\pi}\left[(m_1a_1^{-2}+0.75m_3+0.25m_3\cos\varphi_1)^{1/2}+\varepsilon a_1^{-1}\Phi_{11}\right]^{-1}\mathrm{d}\varphi_1/$$
$$\int_0^{2\pi}\left[\Phi_{20}+\varepsilon\Phi_{21}\right]^{-1}\mathrm{d}\varphi_2 \tag{50}$$

在式 (48) 中

$$\Phi_1^*=\varphi_1+\sum_{n=1}^{M}\frac{1}{2nc_{20}}(c_{1,2n}\sin 2n\varphi_1-c_{2,2n}\sin 2n\varphi_2) \tag{51}$$

现在取参数 $m_1=-1$, $m_3=1$, $k_1=4$, $k_3=-0.5$, $\lambda_1=2$, $\lambda_2=0.5$, $\varepsilon=0.1$ 和 $M=4$, 由式 (48)—(50) 和 (22) 求得

$$a_1=2.332,\quad a_2=1.823,\quad \gamma_2=-0.5918 \tag{52}$$

$$\varphi_1=\varphi_2-0.5928-0.1204\cos 2\varphi_2+0.1057\sin 2\varphi_2$$
$$-0.0099\cos 4\varphi_2+0.0001\sin 4\varphi_2=\Psi_{12}(\varphi_2) \tag{53}$$

$$\varphi_2=\varphi_1+0.5928-0.0527\cos 2\varphi_1-0.1508\sin 2\varphi_1$$
$$+0.0088\cos 4\varphi_1+0.0137\sin 4\varphi_1=\Psi_{21}(\varphi_1) \tag{54}$$

对应的周期解为

$$u_1=2.332\cos\varphi_1\,,\qquad \dot{u}_1=-2.332(\Phi_{10}+\varepsilon\Phi_{11})\sin\varphi_1 \tag{55}$$

$$u_2=1.823\cos\varphi_2\,,\qquad \dot{u}_2=-1.823(\Phi_{20}+\varepsilon\Phi_{21})\sin\varphi_2 \tag{56}$$

其中

$$\Phi_{10}=(3.078+1.359\cos 2\varphi_1)^{1/2} \tag{57}$$

$$\Phi_{20}=(2.753-0.415\cos 2\varphi_2)^{1/2} \tag{58}$$

$$\Phi_{11}=\int_0^{\varphi_1}[(1-5.438\cos^2\varphi_1)\Phi_{10}\sin^2\varphi_1-1.563\cos\varphi_2\sin\varphi_1]\mathrm{d}\varphi_1/\Phi_{10}\sin^2\varphi_1 \tag{59}$$

$$\Phi_{21}=\int_0^{\varphi_2}[(1-3.323\cos^2\varphi_2)\Phi_{20}\sin^2\varphi_2-0.639\cos\varphi_1\sin\varphi_2]\mathrm{d}\varphi_2/\Phi_{20}\sin^2\varphi_2 \tag{60}$$

由式 (55) 和 (56) 作出的相图如图 1 所示，图中还给出了用四阶 Runge-Kutta 数值积分法得到的结果．两者比较表明，用本文渐近方法得到的近似解与用数值法求得的解是相当吻合的．

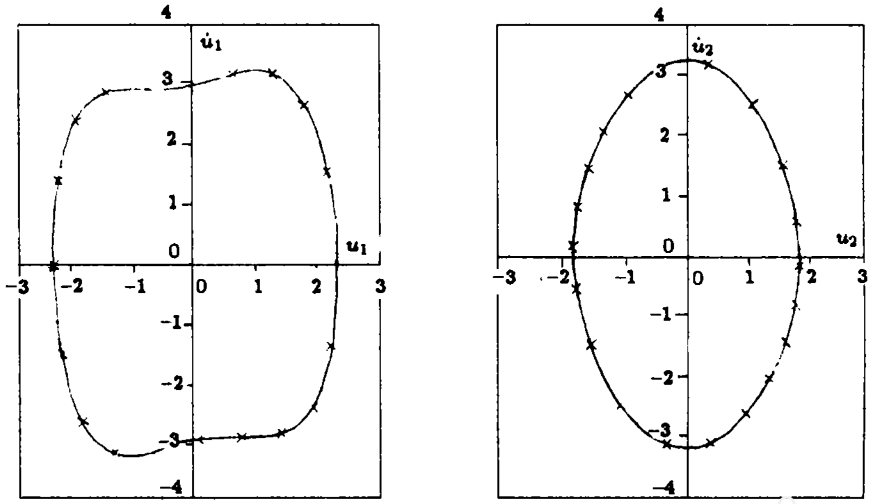

图 1 方程 (35) 的极限环相图， $m_1=-1$, $m_3=1$
$k_1=4$, $k_3=-0.5$, $\lambda_1=2$, $\lambda_2=0.5$, $\varepsilon=0.1$
—— 数值积分法， × 本文方法

Fig.1 Limit cycles of equation (35), for $m_1=-1$, $m_3=1$
$k_1=4$, $k_3=-0.5$, $\lambda_1=2$, $\lambda_2=0.5$, $\varepsilon=0.1$
——Numerical integration method, × The present method

例 2 再考虑有非对称恢复力的情形

$$\left.\begin{aligned}\ddot{u}_1+m_1u_1+m_3u_1^3=\varepsilon(1-u_1^2)\dot{u}_1+\varepsilon\lambda_1u_2\\ \ddot{u}_2+k_1u_2+k_2u_2^2=\varepsilon(1-u_2^2)\dot{u}_2+\varepsilon\lambda_2u_1\end{aligned}\right\} \tag{61}$$

这里 $g_2(u_2)=k_1u_2+k_2u_2^2$ 含有 u_2 的偶次项，属于有非对称恢复力的情形． $v_2(u_2)=\frac{1}{2}k_1u_2^2+\frac{1}{3}k_2u_2^3$．由式 (9) 和 (10) 得

$$b_2=\frac{1}{2k_2}\left(-k_1+\sqrt{k_1^2-\frac{4}{3}k_2^2a_2^2}\right) \tag{62}$$

$$\Phi_{20}(a_2,\varphi_2)=\sqrt{k_1+2k_2b_2+\frac{2}{3}k_2a_2\cos\varphi_2} \tag{63}$$

其余与例 1 相同．现取 $m_1=-1$, $m_3=2$, $k_1=6$, $k_2=1$, $\lambda_1=2$, $\lambda_2=0.5$, $\varepsilon=0.1$．按上例同样方法求得

$$\begin{aligned}&a_1=2.156,\quad a_2=1.889,\quad \gamma_2=-0.3869\\ &b_1=0,\quad b_2=-0.205,\quad x_{11}=-0.082,\quad x_{21}=0.013\end{aligned} \tag{64}$$

$$\begin{aligned}\varphi_1=\varphi_2&-0.3969+0.0074\cos\varphi_2-0.1104\sin\varphi_2\\ &-0.0766\cos2\varphi_2+0.0871\sin2\varphi_2+0.0090\cos3\varphi_2\\ &-0.0088\sin3\varphi_2-0.0047\cos4\varphi_2+0.0008\sin4\varphi_2\end{aligned} \tag{65}$$

$$\varphi_2 = \varphi_1 + 0.3969 + 0.0347\cos\varphi_1 + 0.1110\sin\varphi_1 \\ -0.0054\cos 2\varphi_1 - 0.1105\sin 2\varphi_1 - 0.0021\cos 3\varphi_1 \\ -0.0067\sin 3\varphi_1 + 0.0012\cos 4\varphi_1 + 0.0087\sin 4\varphi_1 \tag{66}$$

对应的周期解为

$$u_1 = 2.156\cos\varphi_1 - 0.0082\ ,\quad \dot u_1 = -2.156(\Phi_{10} + \varepsilon\Phi_{11})\sin\varphi_1 \tag{67}$$

$$u_2 = 1.889\cos\varphi_2 - 0.2037\ ,\quad \dot u_2 = -1.889(\Phi_{20} + \varepsilon\Phi_{21})\sin\varphi_2 \tag{68}$$

其中

$$\Phi_{10} = (5.973 + 2.324\cos 2\varphi_1)^{1/2} \tag{69}$$

$$\Phi_{20} = (5.590 + 1.259\cos\varphi_2)^{1/2} \tag{70}$$

$$\Phi_{11} = \Big\{\int_0^{\varphi_1}[(1 - 4.648\cos^2\varphi_1)\Phi_{10}\sin^2\varphi_1 + (0.190 - 1.752\cos\varphi_2)\sin\varphi_1]\mathrm{d}\varphi_1 \\ -0.316 - 0.037\cos\varphi_1 + 0.353\cos^3\varphi_1\Big\}\Big/\Phi_{10}\sin^2\varphi_1 \tag{71}$$

$$\Phi_{21} = \Big\{\int_0^{\varphi_2}[(0.958 + 0.774\cos\varphi_2 - 3.568\cos^2\varphi_2)\Phi_{20}\sin^2\varphi_2 \\ -0.571\cos\varphi_1\sin\varphi_2]\mathrm{d}\varphi_2 + 0.051 - 0.038\cos\varphi_2 - 0.013\cos^2\varphi_2\Big\}\Big/\Phi_{20}\sin^2\varphi_2 \tag{72}$$

由 (67) 和 (68) 作出的极限环相图如图 2 所示，结果与数值法比较也很吻合.

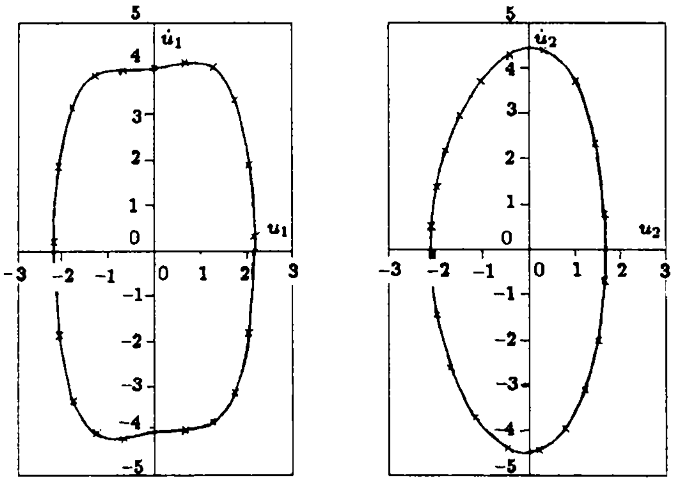

图 2　方程 (61) 的极限环相图，$m_1 = -1$, $m_3 = 2$
$k_1 = 6$, $k_2 = 1$, $\lambda_1 = 2$, $\lambda_2 = 0.5$, $\varepsilon = 0.1$
—— 数值积分法，× 本文方法

Fig.2 Limit cycles of equation (61), for $m_1 = -1$, $m_3 = 2$
$k_1 = 6$, $k_2 = 1$, $\lambda_1 = 2$, $\lambda_2 = 0.5$, $\varepsilon = 0.1$
——Numerical integration method, × The present method

4 结 论

本文对一类多自由度强非线性振动系统提出了渐近解法，它是文 [1] 新方法的进一步发展．应用本方法很容易求出系统的周期解．从实例的分析结果与数值法比较，可以看出本文方法是有效的并有较好的精度．这里所述的方法还可以推广应用于多自由度强非线性受迫振动系统，我们将另文研究．

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ASYMPTOTIC METHOD FOR PRIMARY RESONANCE OF A STRONGLY NONLINEAR VIBRATION SYSTEM WITH MANY DEGREES OF FREEDOM

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Abstract In this paper, an asymptotic method is presented for the analysis of a class of strongly nonlinear autonomous vibrating systems with many degrees of freedom. It can be viwed as a generalization of the new asymptotic method[1]. This method is suitable for the systems in which primary resonance may occur. Using the present method the equations governing the amplitudes and the phase factors are established. Two numerical examples are presented which served to demonstrate the effectiveness of the present method.

Key words strongly nonlinear systems with many degrees of freedom, asymptotic method, periodic solution, primary resonance