具有记忆的污染物两维分散的描述[1)]

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提要　在污染物释放后的前期它的扩散过程受出口速度的影响而具有明显的记忆特性，本文将 R. Smith[8](1981) 污染物纵向记忆描述理论推广到两维情况

$$\frac{\partial \bar{C}}{\partial t}+\bar{u}\frac{\partial \bar{C}}{\partial x}+\bar{v}\frac{\partial \bar{C}}{\partial y}-\bar{K}_{xx}\frac{\partial^2 C}{\partial x^2}-\bar{K}_{yy}\frac{\partial^2 C}{\partial y^2}-\int_0^\infty\Big\{\frac{\partial D_{xx}}{\partial \tau}\frac{\partial^2}{\partial x^2}+\Big[\frac{\partial D_{xy}}{\partial \tau}+\frac{\partial D_{yx}}{\partial \tau}\Big]\frac{\partial^2}{\partial x\partial y}+\frac{\partial D_{yy}}{\partial \tau}\frac{\partial^2}{\partial y^2}\Big\}\bar{C}(x-\int_0^\tau\tilde{u}\mathrm{d}\tau',y-\int_0^\tau\tilde{v}\mathrm{d}\tau',t-z)\mathrm{d}\tau=\bar{q}(x,y,t)$$

并给出了 $\frac{\partial D_{xx}}{\partial \tau}$，$\frac{\partial D_{xy}}{\partial \tau}$，$\frac{\partial D_{yx}}{\partial \tau}$，$\frac{\partial D_{yy}}{\partial \tau}$ 及 $\tilde{u}$，$\tilde{v}$ 的解析表达式.

关键词　剪切分散，污染物扩散，分析解，特征值问题，记忆特性

一、引　　言

自从 Taylor[1−3](1953, 1954) 证明分散在示踪质扩散和输运中起着重要作用以来，大量关于剪切分散的研究工作出现了[4−8]. 正如 Taylor 本人所指出的那样，Taylor 理论只适用于污染物释放后相当长的一段时间以后. 为了弥补这一不足，Gill 和 Sankarasubramanian[5](1970) 提出了变导数扩散方程

$$\frac{\partial \bar{C}}{\partial t}+\hat{u}(t)\frac{\partial \bar{C}}{\partial x}-[k+D(t)]\frac{\partial^2 \bar{C}}{\partial x^2}=0 \tag{1.1}$$

来模拟早期分散过程. 并获得了较准确的污染物中心位移和方差. 但这样处理的物理意义不明确. 正如 Taylor[3](1959) 所强调的：任何数学方式的处理都必须基于一定的物理基础之上. 不具有物理意义的描述显然是逊色的.

为此 R. Smith[8](1981) 指出了对于纵向分散早期可以考虑具有记忆特性的浓度场表达式

$$C-\bar{C}=\sum_{j=1}^\infty\int_0^\infty l_j(y,z,\tau)\frac{\partial^j}{\partial x^j}\bar{C}(x-\int_0^\tau\tilde{u}\mathrm{d}\tau',t-\tau)\mathrm{d}\tau \tag{1.2}$$

其中 C 为浓度，$\bar{C}$ 为浓度的垂向平均，$\tilde{u}$ 被称之为输运速度，l_j 被称之为权函数. 利用这一表达式. Smith 较好地处理了纵向分散早期特性.

本文为了研究河口、湖泊、海湾等情况的二维早期分散特性，将 Smith 的理论推广到两维情况，并且将给出两维的具有记忆特性的剪切分散导数的表达式.

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二、水平与垂向分散方程

一般浓度扩散方程如下

$$\left.\begin{aligned}&\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}-K_{xx}\frac{\partial^2 C}{\partial x^2}-K_{yy}\frac{\partial^2 C}{\partial y^2}-\frac{\partial}{\partial z}K_{zz}\frac{\partial C}{\partial z}=q\\&k\boldsymbol{n}\cdot\nabla C|_{\partial A}=0\end{aligned}\right\}\tag{2.1}$$

对于二维情况，我们建议如下分析式

$$C-\bar{C}=\sum_{m=1}^{\infty}\int_0^{\infty}l_{ij}^m(z,t)\frac{\partial^m}{\partial x^i\partial y^j}\bar{C}\left(x-\int_0^{\tau}\tilde{u}(\tau')\mathrm{d}\tau',y-\int_0^{\tau}\tilde{v}(\tau')\mathrm{d}\tau',(t-\tau)\right)\mathrm{d}\tau\tag{2.2}$$

将 (2.2) 代入 (2.1) 并取垂向平均，得到积分微分方程式

$$\begin{aligned}&\frac{\partial\bar{C}}{\partial t}+\bar{u}\frac{\partial\bar{C}}{\partial x}+\bar{v}\frac{\partial\bar{C}}{\partial y}-\bar{K}_{xx}\frac{\partial^2 C}{\partial x^2}-\bar{K}_{yy}\frac{\partial^2 C}{\partial y^2}+\sum_{m=1}^{\infty}\int_0^{\infty}\Big(\overline{l_{ij}^m(u-\bar{u})}\\&\cdot\frac{\partial^{m+1}}{\partial x^{i+1}\partial y^j}+\overline{l_{ij}^m(v-\bar{v})}\frac{\partial^{m+1}}{\partial x^i\partial y^{j+1}}-\overline{(l_{ij}^m(K_{xx}-\bar{K}_{xx})}\\&\cdot\frac{\partial^{m+2}}{\partial x^{i+2}\partial y^j}+\overline{l_{ij}^m(K_{yy}-\bar{K}_{yy})}\frac{\partial^{m+2}}{\partial x^i\partial y^{j+2}}\Big)\bar{C}\mathrm{d}\tau=\bar{q}(x,y,t)\end{aligned}\tag{2.3}$$

(2.3) 式主要数学特性由 $\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}-\bar{K}_{xx}\frac{\partial^2}{\partial x^2}-\bar{K}_{yy}\frac{\partial^2}{\partial y^2}$ 所确定，我们称 (2.3) 式为水平分散方程．为方便起见，通常令 $m=1$，得到低阶的 (2.3) 分散方程近似

$$\begin{aligned}&\frac{\partial\bar{C}}{\partial t}+\bar{u}\frac{\partial\bar{C}}{\partial x}+\bar{v}\frac{\partial\bar{C}}{\partial y}-\bar{K}_{xx}\frac{\partial^2 C}{\partial x^2}-\bar{K}_{yy}\frac{\partial^2 C}{\partial y^2}-\int_0^{\infty}\Big[\frac{\partial D_{xx}}{\partial\tau}\frac{\partial^2}{\partial x^2}+\frac{\partial D_{yx}}{\partial\tau}\frac{\partial^2}{\partial x\partial y}\\&+\frac{\partial D_{xy}}{\partial\tau}\frac{\partial^2}{\partial x\partial y}+\frac{\partial D_{yy}}{\partial\tau}\frac{\partial^2}{\partial y^2}\Big]\bar{C}\left(x-\int_c^{\tau}\tilde{u}\mathrm{d}\tau,y-\int_0^{\tau}\tilde{v}\mathrm{d}\tau,t-\tau\right)\mathrm{d}\tau=q(x,y,\tau)\end{aligned}\tag{2.4}$$

其中

$$\frac{\partial D_{xx}}{\partial\tau}=\overline{l'_{1,0}(\bar{u}-u)},\qquad\frac{\partial D_{xy}}{\partial\tau}=\overline{l'_{1,0}(\bar{v}-v)}$$
$$\frac{\partial D_{yx}}{\partial\tau}=\overline{l'_{0,1}(\bar{u}-u)},\qquad\frac{\partial D_{yy}}{\partial\tau}=\overline{l'_{0,1}(\bar{v}-v)}$$

现在我们主要是研究 l_{ij}, $\tilde{u}$, $\tilde{v}$, 为了求 l_{ij}, $\tilde{u}$, $\tilde{v}$, 我们写出垂向浓度涨落方程式，将 (2.2) 式代入 (2.1) 并利用 (2.3) 式得到

$$\left.\begin{aligned}&[l'_{1,0}+u-\bar{u}]|_{\tau=0}\frac{\partial\bar{C}}{\partial x}+[l'_{0,1}+v-\bar{v}]|_{\tau=0}\frac{\partial\bar{C}}{\partial y}+[l_{2,0}^2(z,0)+\bar{K}_{xx}-K_{xx}]|_{\tau=0}\\&\cdot\frac{\partial^2\bar{C}}{\partial x^2}+[l_{0,2}^2(z,0)+\bar{K}_{yy}-K_{yy}]|_{\tau=0}\frac{\partial^2 C}{\partial y^2}+\sum_{m=1}^{\infty}\int_0^{\infty}\Big[l_{ij}^m-\tilde{u}l_{i-1,j}^{m-1}-\tilde{v}l_{i,j-1}^{m-1}\\&-\overline{l_{i-1,j}^{m-1}u}-\overline{l_{i,j-1}^{m-1}v}+ul_{i-1,j}^{m-1}+vl_{i,j-1}^{m-1}+\overline{l_{i-2,j}^{m-2}K_{xx}}+\overline{l_{i,j-2}^{m-2}K_{yy}}\\&-l_{i-2,j}^{m-2}K_{xx}-l_{i,j-2}^{m-2}K_{yy}-\frac{\partial}{\partial z}K_{yy}\frac{\partial l_{ij}^m}{\partial z^m}\Big]\frac{\partial^m\bar{C}}{\partial x^i\partial y^j}\mathrm{d}\tau=0\\&\sum_{m=1}^{\infty}\int_0^{\infty}K\boldsymbol{n}\cdot\nabla l_{ij}^m\frac{\partial^m}{\partial x^i\partial y^j}\bar{C}\mathrm{d}\tau|_{\partial A}=0\end{aligned}\right\}\tag{2.5}$$

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并且有

$$l_{ij}^m = 0\,, \qquad \text{如果} \qquad i + j \le 0$$

由于求解过程必须是对任何 $\bar{C}$ 分布都成立，这要求对应的 $\dfrac{\partial^m \bar{C}}{\partial x^i \partial y^j}$ 的导数必须等于零. 由此而导出一系列边初值问题. (精确到两阶)

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau} l'_{1,0} - \frac{\partial}{\partial z} K_{zz} \frac{\partial l'_{1,0}}{\partial z} = 0 \\ &K\boldsymbol{n} \cdot \nabla l'_{1,0}|_{\partial A} = 0\,, \quad l'_{1,0}|_{\tau=0} = \bar{u} - u \end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau} l'_{0,1} - \frac{\partial}{\partial z} K_{zz} \frac{\partial}{\partial z} l'_{0,1} = 0 \\ &K\boldsymbol{n} \cdot \nabla l'_{0,1}|_{\partial A} - 0\,, \quad l'_{0,1}|_{\tau=0} = \bar{u} - u \end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau} l^2_{2,0} - \frac{\partial}{\partial z} K_{zz} \frac{\partial}{\partial z} l^2_{2,0} = \tilde{u} l'_{1,0} + \overline{u l'_{1,0}} - l_{1,0} u \\ &K\boldsymbol{n} \cdot \nabla l^2_{2,0}|_{\partial A} = 0\,, \quad l^2_{2,0}|_{\tau=0} = K_{xx} - \bar{K}_{xx} \end{aligned}\right\} \tag{2.8}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau} l^2_{0,2} - \frac{\partial}{\partial z} K_{zz} \frac{\partial}{\partial z} l^2_{0,2} = \tilde{v} l'_{0,1} + \overline{v l_{0,1}} - v l'_{0,1} \\ &K\boldsymbol{n} \cdot \nabla l^2_{0,2}|_{\partial A} = 0\,, \quad l^2_{0,2}|_{\tau=0} = K_{yy} - \bar{K}_{yy} \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau} l^2_{1,1} - \frac{\partial}{\partial z} K_{zz} \frac{\partial}{\partial z} l^2_{1,1} \\ &= \tilde{u} l'_{0,1} - \tilde{v} l'_{1,0} + (\overline{u l'_{0,1}} - u l'_{0,1}) + (\overline{v l'_{1,0}} - v l'_{1,0}) \\ &K\boldsymbol{n} \cdot \nabla l^2_{1,1}|_{\partial A} = 0\,, \quad l^2_{1,1}|_{\tau=0} = 0 \end{aligned}\right\} \tag{2.10}$$

这些方程组的物理含义在于描述污染物在垂向结构变化. 初值和强迫项则是由于速度场垂向分布不均匀而引起的. 类似于 Gill 和 Sankaraswbramanian[5] 的分析. 可以决定这些函数 $l_{i,j}$ 低阶将起决定性的作用.

三、延迟分散函数

由方程 (2.5)—(2.9) 可知这些定解问题存在一个特征函数 χ_m 和特征值 λ_m, 它满足如下方程

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z} K_{zz} \frac{\mathrm{d}}{\mathrm{d}z} \chi_m + \lambda_m \chi_m = 0 \\ &K\boldsymbol{n} \cdot \nabla \chi_m|_{\partial A} = 0 \end{aligned}\right\} \tag{3.1}$$

特征函数的最低阶 $\lambda_0 = 0$ 对应于 $\chi_0 = 1$, 此时表示沿深度方向的浓度分布均匀状态. 将 u, v 分别用特征函数来表示，定义系数

$$\left.\begin{aligned} u_m &= \overline{(u - \bar{u})\chi_m} / (\overline{\chi_m^2})^{1/2} \\ v_m &= \overline{(v - \bar{v})\chi_m} / (\overline{\chi_m^2})^{1/2} \end{aligned}\right\} \tag{3.2}$$

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于是 (2.6), (2.7) 具有如下形式解

$$\left.\begin{aligned} l'_{1,0} &= -\sum_{m=1}^{\infty} u_m \exp(-\lambda_m\tau)\chi_m(z)/(\overline{\chi_m^2})^{1/2} \\ l'_{0,1} &= -\sum_{m=1}^{\infty} v_m \exp(-\lambda_m\tau)\chi_m(z)/(\overline{\chi_m^2})^{1/2} \end{aligned}\right\} \tag{3.3}$$

于是低阶截断 (2.3) 由下式给出

$$\left.\begin{aligned} \frac{\partial}{\partial\tau}D_{xx} &= \overline{l'_{1,0}(\bar{u}-u)} = \sum_{m=1}^{\infty} u_m^2 \exp(-\lambda_m\tau) \\ \frac{\partial}{\partial\tau}D_{yy} &= \overline{l'_{0,1}(\bar{v}-v)} = \sum_{m=1}^{\infty} v_m^2 \exp(-\lambda_m\tau) \\ \frac{\partial}{\partial\tau}D_{xy} &= \overline{l'_{1,0}(\bar{v}-v)} = \sum_{m=1}^{\infty} u_m v_m \exp(-\lambda_m\tau) \\ \frac{\partial}{\partial\tau}D_{yx} &= \overline{l'_{0,1}(\bar{u}-u)} = \sum_{m=1}^{\infty} v_m u_m \exp(-\lambda_m\tau) \end{aligned}\right\} \tag{3.4}$$

只要给出 D_{xx}, D_{yy}, D_{xy}, D_{yx} 的定解条件，积分上式即可求解.

四、$\bar{u}(\tau)$ 及 $\bar{v}(\tau)$ 的确定

为了求解 (2.8)—(2.10) 引入系数

$$\left.\begin{aligned} u_{mn} &= \overline{(\overline{u\chi_m} - u\chi_m)\chi_n}/\overline{(\chi_m^2)^{1/2}}\,\overline{(\chi_n^2)^{1/2}} \\ v_{mn} &= \overline{(\overline{v}\chi_m - v\chi_m)\chi_n}/\overline{(\chi_m^2)^{1/2}}\,\overline{(\chi_n^2)^{1/2}} \\ K_{xm} &= \overline{(K_{xx} - \bar{K}_{xx})\chi_m}/\overline{(\chi_m^2)^{1/2}} \\ K_{ym} &= \overline{(K_{yy} - \bar{K}_{yy})\chi_m}/\overline{(\chi_m^2)^{1/2}} \end{aligned}\right\} \tag{4.1}$$

于是求得

$$\left.\begin{aligned} l^2_{2,0} &= \sum_{m=1}^{\infty} a_m(\tau)\exp(-\lambda_m\tau)[\chi_m/\overline{(\chi_m^2)}^{1/2}] \\ l^2_{0,2} &= \sum_{m=1}^{\infty} b_m(\tau)\exp(-\lambda_m\tau)[\chi_m/\overline{(\chi_m^2)}^{1/2}] \\ l^2_{1,1} &= \sum_{m=1}^{\infty} c_m(\tau)\exp(-\lambda_m\tau)[\chi_m/\overline{(\chi_m^2)}^{1/2}] \end{aligned}\right\} \tag{4.2}$$

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其中 a_m, b_m, c_m 分别满足如下常微分方程

$$\left.\begin{aligned}
&\frac{\mathrm{d}a_m}{\mathrm{d}\tau} = \sum_{m=1}^{\infty} u_{mn}u_n \exp((\lambda_m - \lambda_n)\tau) - \tilde{u}(\tau)u_m \\
&a_m(0) = K_{xm} \\
&\frac{\mathrm{d}}{\mathrm{d}\tau}b_m = \sum_{n=1}^{\infty} v_{mn}v_n \exp((\lambda_m - \lambda_n)\tau) - \tilde{v}(\tau)v_m \\
&B_m(0) = K_{ym} \\
&\frac{\mathrm{d}}{\mathrm{d}\tau}c_m = \sum_{n=1}^{\infty}[u_{mn}v_n + v_{mn}u_n] \exp((\lambda_m - \lambda_n)\tau) - (\tilde{u}(\tau)v_m + \tilde{v}(\tau)u_m) \\
&c_m(0) = 0
\end{aligned}\right\} \tag{4.3}$$

积分 (4.3) 式得到

$$\left.\begin{aligned}
&a_m = K_{xm} + u_{mn}(u_m\tau - \int_0^\tau \tilde{u}(\tau)\mathrm{d}\tau + \sum_{m\neq n} u_m u_n \exp[((\lambda_m - \lambda_n)\tau) - 1]/(\lambda_m - \lambda_n) \\
&b_m = K_{ym} + v_{mm}(v_m\tau - \int_0^\tau \tilde{v}(\tau)\mathrm{d}\tau) + \sum_{m\neq n} v_{mn}v_n \frac{\exp[(\lambda_m - \lambda_n)\tau - 1]}{\lambda_m - \lambda_n} \\
&c_m = v_m(-\tau u_{mm} + \int_0^\tau \tilde{u}(\tau)\mathrm{d}\tau) + u_m(-v_{mm}\tau - \int_0^\tau \tilde{v}(\tau)\mathrm{d}\tau) \\
&\quad + \sum_{m\neq n}[u_{mn}v_n + v_{mn}u_n]\frac{\exp[(\lambda_m - \lambda_n)\tau] - 1}{\lambda_m - \lambda_n}
\end{aligned}\right\} \tag{4.4}$$

(4.4) 式的解取决于 $\tilde{u}$, $\tilde{v}$ 的选择.

由于 (2.4) 比 (2.3) 更为常用. 故选择 $\tilde{u}$, $\tilde{v}$ 的准则, 就是使 (2.4) 求解更接近于 (2.3) 式的解. 为此我们考虑高阶分散方程, 从方程 (2.5) 式得到对 $\frac{\partial^3\bar{C}}{\partial x^3}$ 与 $\frac{\partial^3 C}{\partial y^3}$ 的分散贡献项

$$\begin{aligned}
&\overline{l'_{1,0}(\bar{K}_{xx} - K_{xx})} + \overline{l^2_{2,0}(u - \bar{u})} \\
&= 2\sum_{m=1}^{\infty} K_{xm}u_m \exp(-\lambda_m\tau) + \sum_{m=1}^{\infty}(u_{mm}\tau - \int_0^\tau \tilde{u}(\tau')\mathrm{d}\tau')u_m^2 \exp(-\lambda_m\tau) \\
&\quad + \sum_{m=1}^{\infty}\sum_{m\neq n} u_n u_m u_{mn}\left\{\frac{\exp(-\lambda_n\tau) - \exp(-\lambda_m\tau)}{\lambda_m - \lambda_n}\right\}
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\overline{l'_{0,1}(\bar{K}_{yy} - K_{yy})} + \overline{l^2_{0,2}(v - \bar{v})} \\
&= 2\sum_{m=1}^{\infty} K_{ym}v_m \exp(-\lambda_m\tau) + \sum_{m=1}^{\infty}(v_{mn}\tau - \int_0^\tau \tilde{v}(\tau')\mathrm{d}\tau')v_m^2 \exp(-\lambda_m\tau) \\
&\quad + \sum_{m=1}^{\infty}\sum_{m\neq n} v_n v_m v_{mn}\left\{\frac{\exp(-\lambda_n\tau) - \exp(-\lambda_m\tau)}{\lambda_m - \lambda_n}\right\}
\end{aligned} \tag{4.6}$$

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(4.5) 和 (4.6) 两式右边的第一项表示扩散系数的垂向变化．而余下的两项则表示比 (3.4) 式更高阶的剪切效应近似．根据 Smith[8] 的建议，最优化的选择是使这种高阶剪切修正近似为零，于是有

$$\left.\begin{aligned}
&\frac{\partial D_{xx}}{\partial \tau}\cdot\int_0^\tau(\tilde{u}(\tau')-\bar{u})\mathrm{d}\tau' \\
&= \tau\sum_{m=1}^{\infty}(u_{mm}-\bar{u})u_m^2\exp(-\lambda_m\tau)+\sum_{m=1}^{\infty}\sum_{n\neq m}u_m u_n u_{mn}\left\{\frac{\exp(-\lambda_n\tau)-\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}\right\} \\
&\frac{\partial}{\partial \tau}D_{yy}\cdot\int_0^\tau(\tilde{v}(\tau')-\bar{v})\mathrm{d}\tau' \\
&= \tau\sum_{m=1}^{\infty}(v_{mm}-\bar{v})v_m^2\exp(-\lambda_m\tau)+\sum_{m=1}^{\infty}\sum_{m\neq n}v_n v_m v_{mn}\left\{\frac{\exp(-\lambda_n\tau)-\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}\right\}
\end{aligned}\right\}\tag{4.7}$$

实际上 (4.7) 式可以直接求出 $\tilde{u}(\tau)$, $\tilde{v}(\tau)$ 的表达式

$$\left.\begin{aligned}
\tilde{u}(\tau)=\;&[\sum_{m=1}^{\infty}u_m^2\exp(-\lambda_m\tau)]^{-2}\Big\{(\sum_{m=1}^{\infty}u_m^2\exp(-\lambda_m\tau))\Big[\sum_{m=1}^{\infty}u_m^2u_{mm}\exp(-\lambda_m\tau) \\
&-\tau\sum_{m=1}^{\infty}\lambda_m u_{mm}u_m^2\exp(-\lambda_m\tau)+\sum_{m=1}^{\infty}\sum_{m\neq n}u_n u_m u_{mn}\frac{\lambda_n\exp(-\lambda_m\tau)-\lambda_n\exp(-\lambda_n\tau)}{\lambda_m-\lambda_n}\Big] \\
&+(\sum_{m=1}^{\infty}u_m^2\lambda_m\exp(-\lambda_m\tau))[\tau\sum_{m=1}^{\infty}u_{mm}u_m^2\exp(-\lambda_m\tau) \\
&+\sum_{m=1}^{\infty}\sum_{m\neq n}u_n u_m u_{mn}\frac{\exp(-\lambda_n\tau)-\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}]\Big\} \\
\tilde{v}(\tau)=\;&[\sum_{m=1}^{\infty}v_m^2\exp(-\lambda_m\tau)]^{-2}\Big\{(\sum_{m=1}^{\infty}v_m^2\exp(-\lambda_m\tau))\Big[\sum_{m=1}^{\infty}v_m^2v_{mn}\exp(-\lambda_m\tau) \\
&-\tau\sum_{m=1}^{\infty}\lambda_m v_{mm}v_m^2\exp(-\lambda_m\tau)+\sum_{m=1}^{\infty}\sum_{m\neq n}v_n v_m v_{mn}\frac{\lambda_m\exp(-\lambda_m\tau)-\lambda_n\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}\Big] \\
&+(\sum_{m=1}^{\infty}v_m^2\lambda_m\exp(-\lambda_m\tau))[\tau\sum_{m=1}^{\infty}v_{mm}v_m^2\exp(-\lambda_m\tau) \\
&+\sum_{m=1}^{\infty}\sum_{m\neq n}v_n v_m v_{mn}\frac{\exp(-\lambda_n\tau)-\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}]\Big\}
\end{aligned}\right\}\tag{4.8}$$

从 (4.8) 式可推得

$$\left.\begin{aligned}
\tilde{u}(0)=\bar{u}+\overline{(u-\bar{u})^3}/\overline{(u-\bar{u})^2}\,,\quad \tilde{u}(\infty)=u_{11} \\
\tilde{v}(0)=\bar{v}+\overline{(v-\bar{v})^3}/\overline{(v-\bar{v})^2}\,,\quad \tilde{v}(\infty)=v_{11}
\end{aligned}\right\}\tag{4.9}$$

事实上，(2.4) 式的正确性取决于 $\tilde{u}$, $\tilde{v}$ 的选择．如果 $K_{xx}=\overline{K_{xx}}$, $K_{yy}=\overline{K_{yy}}$, 那么 (4.7) 式的近似而给出的中心位移、方差、相关矩都是准确的，关于这一点可以通过方程 (2.2) 式乘以 x^iy^j 进行分部积分来验证．

对于 D_{xx}, D_{yx}, D_{xy}, D_{yy} 的确定．我们认为当 $t\to\infty$ 时，污染物 (或示踪质) 的

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记忆特性就肯定消失，那么有

$$\left.\begin{aligned}D_{xx}(\infty)&=-\int(u-\bar u)\mathrm{d}z\int\frac{1}{K_{zz}}\mathrm{d}z\int(u-\bar u)\mathrm{d}z\\D_{xy}(\infty)&=-\int(u-\bar u)\mathrm{d}z\int\frac{1}{K_{zz}}\mathrm{d}z\int(v-\bar v)\mathrm{d}z\\D_{yx}(\infty)&=-\int(v-\bar v)\mathrm{d}z\int\frac{1}{K_{zz}}\mathrm{d}z\int(u-\bar u)\mathrm{d}z\\D_{yy}(\infty)&=-\int(v-\bar v)\mathrm{d}z\int\frac{1}{K_{zz}}\mathrm{d}z\int(v-\bar v)\mathrm{d}z\end{aligned}\right\}\tag{4.10}$$

或者用 Chatwin[4] 的形函数来表示

$$\left.\begin{aligned}D_{xx}(\infty)=\overline{ug}\,,\qquad D_{xy}(\infty)=\overline{uf}\\D_{yx}(\infty)=\overline{vg}\,,\qquad D_{yy}(\infty)=\overline{vf}\end{aligned}\right\}\tag{4.11}$$

其中 f 和 g 满足如下方程

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}K_{zz}\frac{\mathrm{d}g}{\mathrm{d}z}=\bar u-u\\&\frac{\mathrm{d}}{\mathrm{d}z}K_{zz}\frac{\mathrm{d}f}{\mathrm{d}z}=\bar v-v\\&\bar g=\bar f=0\,,\quad k\boldsymbol{n}\cdot\nabla g/\partial A=k\boldsymbol{n}\cdot\nabla f/\partial A=0\end{aligned}\right\}\tag{4.12}$$

很明显，我们所得到的结果：式 (3.4), (4.8), (4.9) 在流场是一维时，即 $v=0$, 则 (3.4), (4.8), (4.9) 退化为纵向延迟扩散解 [R.Smith[8]: (3.4), (4.6), (4.7)].

五、一种近海流中的分散系数

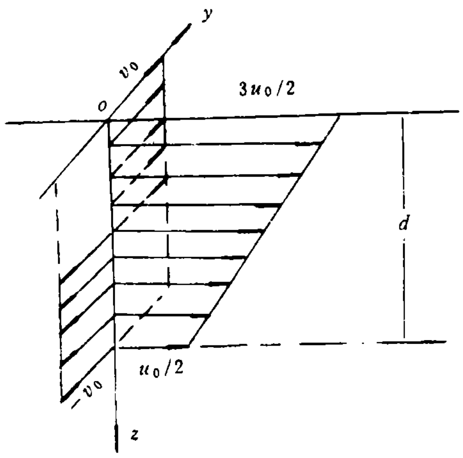

图 1 近海流场简单模型 (Bumpus 1973)

Fig.1 A simple model for shallow water field

我们考虑 Fisher[7](1978) 采用的一种近海流模型．如图 1 示．根据 (4.10) 容易得到

$$D(\infty)=\frac{\mathrm{d}^2}{K_{zz}}\begin{bmatrix}\dfrac{u_0^2}{120}&\dfrac{5u_0v_0}{192}\\\dfrac{5u_0v_0}{192}&\dfrac{v_0^2}{12}\end{bmatrix}\tag{5.1}$$

而特征函数和特征值为

$$\left.\begin{aligned}&\lambda_m=m^2\pi^2K_{zz}/d^2\\&\chi_m=\cos m\pi y/d\\&\chi_m/\overline{(\chi_m^2)}^{1/2}=\sqrt2\cos(m\pi y/d)\end{aligned}\right\}\tag{5.2}$$

由 (3.2) 式， (4.1) 式可得

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$$\left.\begin{aligned}
&u_m=\frac{u_0\sqrt{2}(1-(-1)^m)}{\pi m^2},\quad v_m=(-1)^{m+1}\frac{2\sqrt{2}}{m\pi}v_0\\
&u_{mm}=0,\qquad v_{mm}=0\\
&u_{mn}=\frac{2(m^2+n^2)}{\pi^2(m^2-n^2)^2}(1-(-1)^{m+n})\\
&v_{mn}=\frac{2v_0}{\pi}\Big[\frac{\sin\frac{m+n}{2}\pi}{m+n}+\frac{\sin\frac{m-n}{2}\pi}{m-n}\Big]
\end{aligned}\right\}\tag{5.3}$$

于是

$$\left.\begin{aligned}
&D_{xx}=\frac{1}{120}\frac{u_0^2d^2}{K_{zz}}+\sum_{m=1}^{\infty}\frac{u_0^2d^2}{K_{zz}}\cdot\frac{2(1-(-1)^m)}{\pi^4m^6}\exp\Big(-\frac{\pi^2m^2K_{zz}}{d^2}\tau\Big)\\
&D_{yx}=D_{xy}=\frac{5}{192}\frac{v_0u_0d^2}{K_{zz}}+\sum_{m=1}^{\infty}\frac{u_0v_0d^2}{K_{zz}}\cdot\frac{4(1-(-1)^{m+1}+1)}{\pi^4m^5}\exp\Big(-\frac{\pi^2m^2K_{zz}}{d^2}\tau\Big)\\
&D_{yy}=\frac{1}{12}\frac{v_0^2d^2}{K_{zz}}+\sum_{m=1}^{\infty}\frac{v_0^2d^2}{K_{zz}}\frac{8}{\pi^4m^4}\exp\Big(-\frac{\pi^2K_{zz}}{d^2}m^2\tau\Big)
\end{aligned}\right\}\tag{5.4}$$

图 2 为图 1 简单模型下，剪切分散导数的分布．由 (4.8) 我们可以得到在此模型下，$\bar{u}(\tau)=u_0$, $\bar{v}(\tau)=0$. 这说明在整个分散过程中，污染物团块中心，以平均流速 u_0 沿 x 方向移动．而浓度分布在任一时刻基本上都对应于此时的主轴，但不同时刻其主轴是要变化的．其演变过程如图 3 所示．当 $\frac{K_{zz}}{d^2}\tau=1.75$ 时，基本上与 Fisher(1978) 的定常解结果相吻合．

一般情况下，$\bar{u}$, $\bar{v}$ 不为常数．例如，在上例中，x 方向的速度取为二次函数分布

$$u(z)=\frac{3}{2}\bar{u}\Big[1-\Big(\frac{z}{d}\Big)^2\Big]\tag{5.5}$$

则根据以上相同的步骤，可以得到

$$\left.\begin{aligned}
&u_m=(-1)^{m+1}3\sqrt{2}\bar{u}/m^2\pi^2\\
&u_{mm}=\bar{u}(1-3/\Delta m^2\pi^2)\\
&u_{mn}=\frac{6(m^2+n^2)}{(m^2-n^2)^2}(-1)^{m+n+1}\frac{\bar{u}}{\pi^2}
\end{aligned}\right\}\tag{5.6}$$

$$\left.\begin{aligned}
&D_{xx}=\frac{2}{105}\frac{\bar{u}^2d^2}{K_{zz}}-\frac{18\bar{u}^2d^2}{\pi^6}\sum\frac{1}{m^6}\exp(-m^2\pi^2K_{zz}\tau/d^2)\\
&D_{xy}=D_{yx}=\frac{185}{16}\frac{\bar{u}^2d^2}{K_{zz}}-\frac{z\bar{u}^2d^2}{\pi^5}\sum\frac{1}{m^5}\exp(-m^2\pi^2K_{zz}\tau/d^2)\\
&D_{yy}=\frac{1}{12}\frac{v_0^2d^2}{K_{zz}}-\sum\frac{v_0^2d^2}{K_{zz}}\cdot\frac{8}{\pi^2m^4}\exp\Big(-\frac{\pi^2K_{zz}m^2}{d^2}\tau\Big)
\end{aligned}\right\}\tag{5.7}$$

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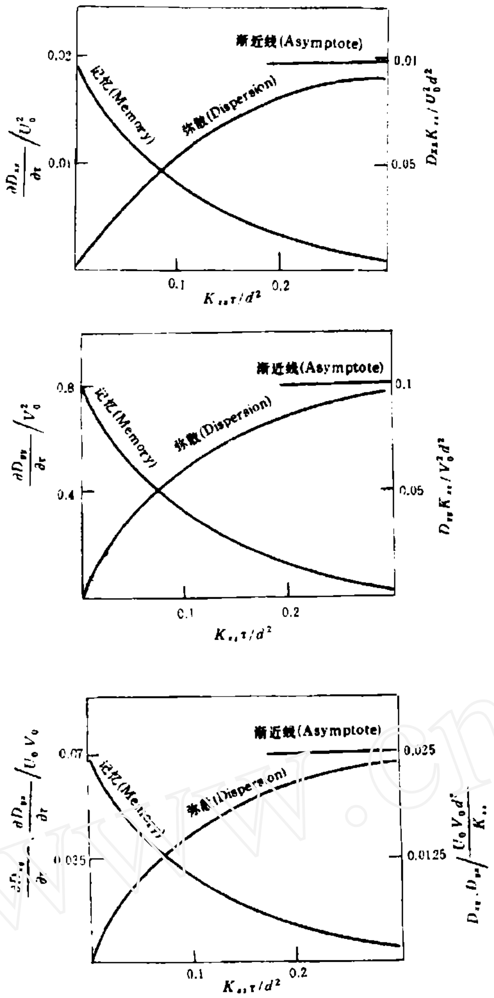

图 2 简单浅海模型下，
记忆函数和剪切弥散系数
Fig.2 Memory function and shear
dispersion for the simple model

此时 D_{ij} 的分布图与图 2 相似. 将 (5.6) 式代入到 (4.8), 可得输运速度分布

$$\left.\begin{aligned}
\tilde{u} = &\Big(\frac{\partial D_{xx}}{\partial\tau}\Big)^{-2}\Big\{\frac{\partial D_{xx}}{\partial\tau}\Big[-\sum_{m=1}^{\infty}u_m^2 u_{mm}\exp(-\lambda_m\tau)-\tau\sum_{m=1}^{\infty}\lambda_m u_{mm}u_m^2\exp(-\lambda_m\tau)\\
&+\sum_{m=1}^{\infty}\sum_{m\neq n}u_m u_{mn}\frac{\lambda_m\exp(-\lambda_m\tau)-\lambda_n\exp(-\lambda_n\tau)}{\lambda_m-\lambda_n}\\
&+\sum_{m=1}^{\infty}\lambda_m u_m^2\exp(-\lambda_m\tau)[\tau\sum_{m=1}^{\infty}u_{mm}u_m\exp(-\lambda_m\tau)]\\
&+\sum_{m=1}^{\infty}\sum_{n\neq m}\frac{\exp(-\lambda_n\tau)-\exp(-\lambda_m\tau)}{\lambda_m-\lambda_n}\Big]\Big\}\\
\tilde{v}(\tau) = &\,0
\end{aligned}\right\} \tag{5.8}$$

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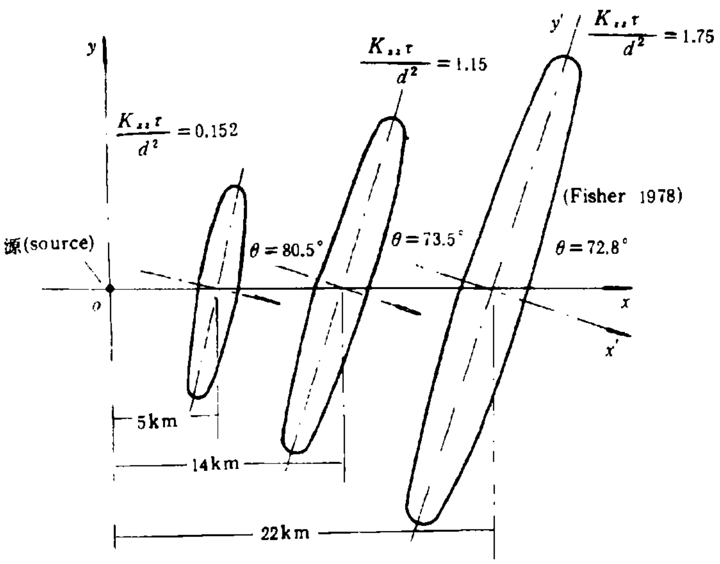

图 3　污染云团在剪切流动中的演变过程，其中各参数值为：

$u_0 = 5\text{cm/s}$, $v_0 = 5\text{cm/s}$, d=50m, K_{zz}=100cm^2/s, $\theta = \frac{1}{2}\text{arctg}\,\frac{D_{yy}-D_{xx}}{D_{xy}(\text{or}D_{yx})}$

Fig.3 Shape of contaminant cloud during the dispersion process

$u_0 = 5\text{cm/s}$, $v_0 = 5\text{cm/s}$, d=50m, K_{zz}=100cm^2/s, $\theta = \frac{1}{2}\text{arctg}\,\frac{D_{yy}-D_{xx}}{D_{xy}(\text{or}D_{yx})}$

图 4 为 $\tilde{u}$ 的分布特征. 从中可以看到，此时在最初污染物团块产生一个很大的负的偏斜率. 这主要是由于在底部 x 方向速度很小所引起的. 为此我们可以得知一般情况下污染物团块的分布不但其 z 轴要转动，而且其高斯分布特性也要改变.

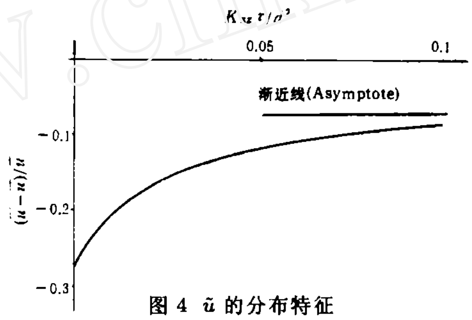

图 4　$\tilde{u}$ 的分布特征

$\tilde{u}(0)-\bar{u}=-\frac{2}{7}\bar{u}, \tilde{u}(\infty)-\bar{u}=-3\bar{u}/4\pi^2$

Fig.4 Distribution of

$\tilde{u}(0)-\bar{u}=-\frac{2}{7}\bar{u}, \tilde{u}(\infty)-\bar{u}=-3\bar{u}/4\pi^2$

六、结论和讨论

对于给定的流动情况，要导出延迟分散方程式 (2.4) 可以按如下步骤：

(1) 决定速度场的分布，并且确定 λ_m 和 χ_m.

(2) 由 (3.4) 确定延迟分散导数.

(3) 导出 u_m, v_m, u_{mn}, v_{mn}. 并且由 (4.8) 式确定 $\tilde{u}(\tau)$ 和 $\tilde{v}(\tau)$

一般根据 $\tilde{u}-\bar{u}$, $\tilde{v}-\bar{v}$ 的正负号可以定性地分析记忆特性的强弱以及对中心位移和方差的影响，当然对于 Thacker[6](1976) 提出的扩散 —— 电极方程也可在两维情形中进行推广. 而得到近似分析记忆特性对中心位移和方差的影响的方程. 有兴趣的读者可以参阅文献 [8] 和 [7].

本研究得到蔡树棠教授和王道坛高级工程师的指导和帮助，在此谨致谢意.

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A DELAY-DIFFUSION DESCRIPTION FOR TWO-DIMENSIONAL CONTAMINANT DISPERSION

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Abstract In the earlier stage of dispersion process, the rate of dispersion associated with earlier discharge is relatively large because the memory term extends further back in time. In order to investigate the early properties of two-dimensional dispersion, we propose the delay-diffusion equation as following:

$$\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} + \bar{V}\frac{\partial \bar{C}}{\partial y} - \bar{K}_{xx}\frac{\partial^2 \bar{C}}{\partial x^2} - \bar{K}_{yy}\frac{\partial^2 \bar{C}}{\partial y^2} - \int^{\infty} \left\{ \frac{\partial D_{xx}}{\partial \tau}\frac{\partial^2}{\partial x^2} + \left[\frac{\partial D_{xy}}{\partial \tau} + \frac{\partial D_{yx}}{\partial \tau}\right]\frac{\partial^2}{\partial x \partial y} + \frac{\partial D_{yy}}{\partial \tau}\frac{\partial^2}{\partial y^2}\right\} \bar{C}\left(x - \int^{\tau} \tilde{u},\ y - \int^{\tau} \tilde{v},\ t - \tau\right) \mathrm{d}\tau = \bar{q}$$

which is based on the ansatz derived by Smith (1981). The analytical expressions for D_{xx}, D_{yx}, D_{xy}, D_{yy} and $\tilde{u}$, $\tilde{v}$ are obtained.

Key words shear dispersion, contaminant dispersion, analytical solution, eigenvalue problem, memory

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