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挠率对螺旋管道内二次流的二阶效应

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提要 本文用螺旋坐标系和摄动法研究了在螺旋管道内的低雷诺数（$R<17/\sqrt{8}$）不可压缩流体的定常粘性流动，给出了完全的二阶摄动解．结果表明管道挠率在二阶摄动解时对二次涡有偏转和扭曲效应．本文还发现当雷诺数从很小值逐渐增大时，两次涡位置发生由左右相对到上下相对的有趣旋转．

关键词 螺旋管流，二次流，摄动法

1. 引言

研究弯曲管道内的流体流动在生物流体力学和化工传递过程的研究中有着重要的意义[1]．但是大部份的研究都是对平面弯管进行的，对挠率不为零的螺旋弯管的研究只有为数不多的报道[2-4]．本文用摄动法对螺旋弯管内充分发展的流动作进一步的研究，求出二阶摄动解，并讨论弯管的曲率和挠率对流动的影响．

螺旋管道的中心线由下列螺线方程表达

$$\boldsymbol{R}_c = c\cos\frac{s}{\sqrt{b^2+c^2}}\boldsymbol{i} + c\sin\frac{s}{\sqrt{b^2+c^2}}\boldsymbol{j} + \frac{bs}{\sqrt{b^2+c^2}}\boldsymbol{k} \tag{1}$$

这里 b,c 为常参数，s 是沿中心线的弧长，$\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ 是直角坐标系中的基矢．中心弧线的切向，法向和次法向单位矢量可分别表达如下

$$\boldsymbol{T}_c = \frac{d\boldsymbol{R}_c}{ds} \quad \boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}_c}{ds} \quad \boldsymbol{B} = \boldsymbol{T}_c\times\boldsymbol{N} \tag{2}$$

由 Frenet 公式

$$\frac{d\boldsymbol{N}}{ds} = \tau\boldsymbol{B} - \kappa\boldsymbol{T}_c \quad \frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N} \tag{3}$$

这里 $\kappa = \frac{c}{b^2+c^2}$，$\tau = \kappa\frac{b}{c}$ 是曲线的曲率和挠率．过中心弧线上的任意一点 s，作一垂直于弧线的平面，平面对弯管的横截面为一半径为 a 的圆，圆内的任意点可由平面极坐标 r,θ 表示，这样我们就构造了一个曲线非正交坐标系 $(x^1,x^2,x^3)=(r,\theta,s)$．半径为 a 的螺旋弯管内的任意空间点可由下述空间矢量表示

$$\boldsymbol{R} = \boldsymbol{R}_c(s) + r\cos\theta\boldsymbol{N}(s) + r\sin\theta\boldsymbol{B}(s) \qquad \begin{matrix} 0\leqslant r\leqslant a \\ 0\leqslant\theta\leqslant 2\pi \end{matrix} \tag{4}$$

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由此可求得该曲线坐标系的度量张量 g_{ij} 和 g^{ij} 的各个非零分量

$$g_{11}=g^{11}=1\quad g_{22}=r^2\quad g^{22}=\frac{G}{r^2M}\quad g_{33}=G$$

$$g^{33}=1/M\quad g_{23}=\tau r^2\quad g^{23}=\frac{-\tau}{M}\tag{5}$$

$$M=(1-\kappa r\cos\theta)^2\quad G=M+\tau^2r^2$$

和相应的非零的 Christoffel 符号

$$\Gamma^1_{22}=-r\quad \Gamma^1_{23}=-\tau r\quad \Gamma^1_{33}=-\frac{1}{2}\frac{\partial G}{\partial r}\quad \Gamma^2_{21}=\frac{1}{r}$$

$$\Gamma^2_{13}=\frac{\tau}{M}\left(\frac{G}{r}-\frac{1}{2}\frac{\partial G}{\partial r}\right)\quad \Gamma^2_{23}=\frac{-\tau}{2M}\frac{\partial G}{\partial\theta}\quad \Gamma^2_{33}=\frac{-G}{2Mr^2}\frac{\partial G}{\partial\theta}$$

$$\Gamma^3_{13}=\frac{-r\tau^2}{M}+\frac{1}{2M}\frac{\partial G}{\partial r}\quad \Gamma^3_{23}=\frac{1}{2M}\frac{\partial G}{\partial\theta}\quad \Gamma^3_{33}=\frac{\tau}{2M}\frac{\partial G}{\partial\theta}\tag{6}$$

在该坐标系中连续方程和纳维-斯托克斯方程取如下形式

$$\frac{\partial V^i}{\partial x^i}+V^i\Gamma^j_{ij}=0\tag{7}$$

$$\frac{\partial V^l}{\partial t}+V^i\frac{\partial V^l}{\partial x^i}+V^iV^j\Gamma^l_{ij}=-g^{kl}\frac{\partial P}{\partial x^k}+\nu g^{mn}$$
$$\cdot\left(\frac{\partial^2V^l}{\partial x^m\partial x^n}+\frac{\partial V^i}{\partial x^n}\Gamma^l_{im}+\frac{\partial V^i}{\partial x^m}\Gamma^l_{ni}-\Gamma^i_{nm}\frac{\partial V^l}{\partial x^i}\right.$$
$$\left.+V^i\frac{\partial\Gamma^l_{ni}}{\partial x^m}+V^i\Gamma^j_{in}\Gamma^l_{jm}-V^i\Gamma^l_{ij}\Gamma^j_{mn}\right)\tag{8}$$

其中 $V^l(l=1,2,3)$ 是速度矢量 $\boldsymbol{V}$ 在该坐标中的逆变分量，即 $\boldsymbol{V}=V^1\boldsymbol{e}_1+V^2\boldsymbol{e}_2+V^3\boldsymbol{e}_3$，$\langle\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\rangle$ 则是坐标系的协变基矢．由于本文仅考虑充分发展的定常流动，则

$$\frac{\partial V^l}{\partial t}=\frac{\partial V^l}{\partial s}=0.$$

2. 纳维-斯托克斯方程的摄动解

通常 $a\kappa\ll 1$，于是令 $\varepsilon\equiv a\kappa$ 为摄动参数，这时 $\tau a=\alpha\varepsilon$ 也是小量，$\alpha\equiv\frac{b}{c}$ 是量级为一的常数．设流体速度和压力有下列形式的摄动解

$$\begin{aligned}V^i&=V^i_0+V^i_1\varepsilon+V^i_2\varepsilon^2+\cdots\quad i=1,2,3\\ P&=P_0(s)+P_1(\theta,r)\varepsilon+P_2(\theta,r)\varepsilon^2+\cdots\end{aligned}\tag{9}$$

并引入下列无量纲变量

$$\left.\begin{aligned}&\xi=\frac{s}{a}\quad \eta=\frac{\gamma}{a}\quad R=\frac{U_a}{\nu}\quad u=\frac{V'}{U}\\ &v=\frac{aV^2}{U}\quad w=\frac{V^3}{U}\quad P=\frac{P_0R}{\rho U^2}\end{aligned}\right\}\tag{10}$$

其中速度标尺U定义为$U \equiv \frac{a}{4\mu}\left(-\frac{dP_0}{d\xi}\right)$，由于采用了非正交归一系，$V^2$ 的量纲是$\frac{1}{[秒]}$. 于是式(9)也可以写成

$$\left.\begin{aligned} u &= u_0 + u_1\varepsilon + u_2\varepsilon^2 + \cdots \\ v &= v_0 + v_1\varepsilon + v_2\varepsilon^2 + \cdots \\ w &= w_0 + w_1\varepsilon + w_2\varepsilon^2 + \cdots \\ p &= p_0 + p_1\varepsilon + p_2\varepsilon^2 + \cdots \end{aligned}\right\} \tag{11}$$

运用式(10)和(11)把方程(7)和(8)无量纲化，再归并 ε 的同次幂项，则不难得到关于 u_i，v_i，w_i 和 p_i 的线性偏微分方程组．零次幂的解马上可以求得，即为

$$u_0 = v_0 = 0 \quad w_0 = (1-\eta^2) \quad p_0 = -4\xi + \cos nt \tag{12}$$

这就是直管中的 Poiseulle 流动的解。一次项的方程则为

$$\frac{\partial u_1}{\partial \eta} + \frac{\partial v_1}{\partial \theta} + \frac{u_1}{\eta} = 0 \tag{13}$$

$$\frac{\partial p_1}{\partial \eta} = \left(\frac{\partial^2 u_1}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u_1}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2 u_1}{\partial \theta^2} - \frac{u_1}{\eta^2} - \frac{2}{\eta}\frac{\partial v_1}{\partial \theta}\right) - Rw_0^2\cos\theta \tag{14}$$

$$\begin{aligned} \frac{\partial p_1}{\partial \theta} = &\left(\eta^2\frac{\partial^2 v_1}{\partial \eta^2} + 3\eta\frac{\partial v_1}{\partial \eta} + \frac{\partial^2 v_1}{\partial \theta^2} + \frac{2}{\eta}\frac{\partial u_1}{\partial \theta}\right) + Rw_0^2\eta\sin\theta \\ &+ 2\alpha\eta\frac{\partial w_0}{\partial \eta} + \alpha\eta^2\frac{\partial p_0}{\partial \xi} \end{aligned} \tag{15}$$

$$\begin{aligned} Dw_1 &\equiv \left(\frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)w_1 \\ &= 3\cos\theta\frac{\partial w_0}{\partial \eta} - 2\eta\cos\theta\frac{\partial p_0}{\partial \xi} - Ru_1\frac{\partial w_0}{\partial \eta} \end{aligned} \tag{16}$$

引进流函数 ψ_1，使

$$u_1 = \frac{1}{\eta}\frac{\partial \psi_1}{\partial \theta} \qquad v_1 = -\frac{1}{\eta}\frac{\partial \psi_1}{\partial \eta} \tag{17}$$

由方程(15)，(16)可得 ψ_1 的如下方程

$$\begin{aligned} L\psi_1 \equiv &\left(\eta^4\frac{\partial^4}{\partial \eta^4} + 2\eta^3\frac{\partial^3}{\partial \eta^3} - \eta^2\frac{\partial^2}{\partial \eta^2} + \eta\frac{\partial}{\partial \eta} + 2\eta^2\frac{\partial^4}{\partial \eta^2\partial \theta}\right. \\ &\left. - 2\eta\frac{\partial^3}{\partial \eta\partial \theta^2} + 4\frac{\partial^2}{\partial \theta^2} + \frac{\partial^4}{\partial \theta^4}\right)\psi_1 \\ = &\, 2\eta^4 w_0\frac{\partial w_0}{\partial \eta}\sin\theta + 4\alpha\eta^3\frac{\partial p_0}{\partial \xi} \end{aligned} \tag{18}$$

此方程满足边界条件

$$\begin{aligned} &u_1 = v_1 = 0 \quad 当\ \eta = 1\ 时 \\ &|u_1| < +\infty \quad |v_1\eta| < +\infty \quad 当\ \eta = 0\ 时 \\ &p_1(\theta) = p_1(\theta + 2\pi) \end{aligned} \tag{19}$$

的解为

$$\psi_1 = \frac{R\eta\sin\theta}{288}(\eta^6 - 6\eta^4 + 9\eta^2 - 4) - \frac{\alpha}{4}(1-\eta^2)^2 \tag{20}$$

进而由方程(14—17)得

$$u_1 = \frac{R\cos\theta}{288}(\eta^6 - 6\eta^4 + 9\eta^2 - 4) \tag{21}$$

$$v_1 = \frac{R\sin\theta}{288}\left(\frac{4}{\eta} - 27\eta + 30\eta^3 - 7\eta^5\right) - \alpha(1-\eta^2) \tag{22}$$

$$p_1 = -\frac{R\cos\theta}{12}(9\eta + 2\eta^5 - 6\eta^3) \tag{23}$$

$$w_1 = \frac{7}{4}\cos\theta(\eta - \eta^3) - \frac{R^2\cos\theta}{11520}(19\eta - 40\eta^3 + 30\eta^5 + 10\eta^7 + \eta^9) \tag{24}$$

二次项的方程则如下

$$\frac{\partial u_2}{\partial\eta} + \frac{\partial v_2}{\partial\theta} + \frac{u_2}{\eta} - u_1\cos\theta + v_1\eta\sin\theta = 0 \tag{25}$$

$$\begin{aligned}\frac{\partial p_2}{\partial\eta} = &-R\left(u_1\frac{\partial u_1}{\partial\eta} + v_1\frac{\partial u_1}{\partial\theta} + 2w_0w_1\cos\theta - w_0^2\eta\cos^2\theta - \eta\alpha^2w_0^2\right.\\ &\left.- v_1^2\eta - 2\alpha\eta w_0v_1\right) + \left(\frac{\partial^2u_2}{\partial\eta^2} + \frac{1}{\eta^2}\frac{\partial u_2}{\partial\theta^2}\right.\\ &\left.+\frac{1}{\eta}\frac{\partial u_2}{\partial\eta} - \frac{u_2}{\eta^2} - \frac{2}{\eta}\frac{\partial v_2}{\partial\theta}\right) - 2\alpha\eta\frac{\partial w_1}{\partial\theta}\\ &+\frac{\sin\theta}{\eta}\frac{\partial u_1}{\partial\theta} - \cos\theta\frac{\partial u_1}{\partial\eta} - v_1\sin\theta\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial p_2}{\partial\theta} = &-R\left(\eta^2u_1\frac{\partial v_1}{\partial\eta} + \eta^2v_1\frac{\partial v_1}{\partial\theta} + 2\eta u_1v_1 + 2\alpha\eta u_1w_0 + \eta^2w_0^2\cos\theta\sin\theta\right.\\ &\left.- 2w_0w_1\eta\sin\theta\right) + \left(\eta^2\frac{\partial^2v_2}{\partial\eta^2} + \frac{\partial^2v_2}{\partial\theta^2}\right.\\ &\left.+\frac{2}{\eta}\frac{\partial u_2}{\partial\theta} + 3\eta\frac{\partial v_2}{\partial\eta}\right) + 2\alpha\eta^3\cos\theta\frac{\partial p_0}{\partial\xi} + u_1\sin\theta\\ &+\eta\sin\theta\frac{\partial v_1}{\partial\theta} - \cos\theta\cdot\eta^2\frac{\partial v_1}{\partial\eta} - \eta\cos\theta v_1\end{aligned} \tag{27}$$

$$\begin{aligned}Dw_2 = &\left(u_1\frac{\partial w_1}{\partial\eta} + v_1\frac{\partial w_1}{\partial\theta}\right)\\ &+u_2\frac{\partial w_0}{\partial\eta} - 2u_1w_0\cos\theta + 2v_1w_0\eta\sin\theta + \alpha\eta w_0^2\sin\theta\end{aligned}$$

$$+3\eta^2\cos^2\theta\frac{\partial p_0}{\partial\xi}-\alpha\frac{\partial p_1}{\partial\theta}+3\eta\cos^2\theta\frac{\partial w_0}{\partial\eta}$$

$$-\frac{3\sin\theta}{\eta}\frac{\partial w_1}{\partial\theta}+3\cos\theta\frac{\partial w_1}{\partial\eta} \tag{28}$$

引进流函数 ψ_2 使

$$u_2=\frac{1}{\eta}\frac{\partial\psi_2}{\partial\theta}+\psi_1\sin\theta \qquad v_2=-\frac{1}{\eta}\frac{\partial\psi_2}{\partial\eta}+\frac{1}{\eta}\psi_1\cos\theta \tag{29}$$

再在方程(26),(27)中消掉压力项,得 ψ_2 的方程

$$L\psi_2=-80\alpha\eta^5\cos\theta+\frac{\alpha R^2\cos\theta}{72}(10\eta^{11}-36\eta^9+36\eta^7-10\eta^5)$$

$$+\frac{R\sin 2\theta}{288}(1632\eta^8-1728\eta^6) \tag{30}$$

$$+\frac{R^3\sin 2\theta}{(288)^2}(600\eta^{14}-806.4\eta^{12}+1080\eta^{10}-1538\eta^8+777.6\eta^6)$$

此方程满足类似于(19)的边界条件的解为

$$\psi_2=-4\alpha\cos\theta\left(\frac{5}{48}\eta-\frac{5}{24}\eta^3+\frac{20}{192}\eta^5\right)$$

$$+\frac{\alpha R^2}{(288)^2}\left(\frac{6}{5}\eta^{11}-\frac{54}{5}\eta^9+36\eta^7-60\eta^5\right.$$

$$\left.+\frac{246}{5}\eta^3-\frac{78}{5}\eta\right)+\frac{R\sin 2\theta}{288}\left(\frac{17}{20}\eta^8-\frac{9}{2}\eta^6\right.$$

$$\left.+\frac{129}{20}\eta^4-\frac{14}{5}\eta^2\right)+\frac{R^3\sin 2\theta}{(288)^2}\left(\frac{5}{244}\eta^{14}-\frac{6}{100}\eta^{12}\right.$$

$$\left.+\frac{3}{16}\eta^{10}-\frac{769}{960}\eta^8+\frac{81}{40}\eta^6-\frac{4997}{2240}\eta^4+\frac{7199}{8400}\eta^2\right) \tag{31}$$

$$u_2=\frac{2\alpha\sin\theta}{3}(1-\eta^2)^2-\frac{\alpha R^2\sin\theta}{288}\left(\frac{\eta^{10}}{240}-\frac{3}{80}\eta^8\right.$$

$$\left.+\frac{1}{8}\eta^6-\frac{5}{24}\eta^4+\frac{41}{240}\eta^2-\frac{13}{240}\right)$$

$$+\frac{R\sin^2\theta}{288}(\eta^7-6\eta^5+9\eta^3-4\eta)+\frac{R\cos 2\theta}{288}\left(\frac{17}{10}\eta^7\right.$$

$$\left.-9\eta^5+\frac{129}{10}\eta^3-\frac{56}{10}\eta\right)+\frac{R^3\cos 2\theta}{(288)^2}\left(\frac{5}{112}\eta^{13}\right.$$

$$\left.-\frac{3}{25}\eta^{11}+\frac{3}{8}\eta^9-\frac{769}{480}\eta^7+\frac{81}{20}\eta^5-\frac{4997}{1120}\eta^3+\frac{7199}{4200}\eta\right) \tag{32}$$

$$v_2=\frac{\alpha\cos\theta}{3\eta}(7\eta^4-9\eta^2+2)-\frac{\alpha R^2\cos\theta}{288}\left(\frac{11}{240}\eta^9-\frac{27}{80}\eta^7\right.$$

$$+\frac{7}{8}\eta^5-\frac{25}{24}\eta^3+\frac{41}{80}\eta-\frac{13}{240}\frac{1}{\eta}\Big)-\frac{R\sin 2\theta}{288}\Big(\frac{63}{10}\eta^6$$

$$-24\eta^4+\frac{213}{10}\eta^2-\frac{36}{10}\Big)-\frac{R^3\sin 2\theta}{(288)^2}\Big(\frac{5}{16}\eta^{12}$$

$$-\frac{18}{25}\eta^{10}+\frac{15}{8}\eta^8-\frac{769}{120}\eta^6+\frac{243}{20}\eta^4-\frac{4997}{560}\eta^2+\frac{7199}{4200}\Big) \tag{33}$$

$$w_2=-\frac{R^4}{(288)^2}\Big(\frac{\eta^{16}}{1280}-\frac{\eta^{14}}{70}+\frac{297}{2880}\eta^{12}-\frac{157}{400}\eta^{10}+\frac{569}{640}\eta^8$$

$$-\frac{99}{80}\eta^6+\frac{331}{320}\eta^4-\frac{38}{80}\eta^2+\frac{4119}{44800}\Big)$$

$$+\frac{R^2}{288}\Big(-\frac{17}{800}\eta^{10}+\frac{7}{32}\eta^8-\frac{5}{8}\eta^6+\frac{23}{32}\eta^4$$

$$-\frac{17}{160}\eta^2-\frac{37}{200}\Big)-\Big(\frac{39}{32}\eta^4-\frac{21}{16}\eta^2+\frac{3}{32}\Big)$$

$$-\frac{R^4\cos 2\theta}{(288)^2}\Big(\frac{\eta^{16}}{2205}-\frac{9}{3200}\eta^{14}+\frac{87}{5600}\eta^{12}-\frac{293}{4608}\eta^{10}$$

$$+\frac{417}{2400}\eta^8-\frac{1033}{3584}\eta^6+\frac{13243}{50400}\eta^4-\frac{1151}{11760}\eta^2\Big)$$

$$+\frac{R^2\cos 2\theta}{288}\Big(-\frac{104}{1920}\eta^{10}+\frac{57}{120}\eta^8-\frac{423}{320}\eta^6+\frac{389}{240}\eta^4$$

$$-\frac{2073}{2880}\Big)+\cos 2\theta\Big(\frac{19}{16}\eta^2-\frac{57}{48}\eta^4\Big)+\alpha R\sin\theta\Big(-\frac{1}{64}\eta^7$$

$$+\frac{5}{72}\eta^5-\frac{\eta^3}{6}+\frac{65}{576}\Big)+\frac{\alpha R^3\sin\theta}{288}\Big(\frac{\eta^{13}}{5040}$$

$$-\frac{7}{2400}\eta^{11}+\frac{1}{64}\eta^9-\frac{13}{210}\eta^7+\frac{109}{1440}\eta^5-\frac{7}{96}\eta^3+\frac{1553}{33600}\Big) \tag{34}$$

$$p_2=\alpha\sin\theta\Big(\frac{\eta^3}{2}-\frac{1}{6}\eta\Big)+\frac{\alpha R^2\sin\theta}{288}\Big(\frac{\eta^9}{20}-\frac{1}{2}\eta^7+\frac{3}{2}\eta^5$$

$$-2\eta^3+\frac{101}{60}\eta\Big)-\frac{R\cos 2\theta}{288}\Big(48\eta^6-132\eta^4$$

$$+\frac{648}{5}\eta^2\Big)+\frac{R^3\cos 2\theta}{(288)^2}\Big(77\frac{353}{560}\eta^2-\frac{894}{5}\eta^4$$

$$+198\eta^6-108\eta^8+\frac{162}{5}\eta^{10}-\frac{234}{35}\eta^{12}\Big)$$

$$-\frac{R}{288}(56\eta^6-162\eta^4+162\eta^2)-\frac{R^3}{(288)^2}\Big(\frac{39}{10}\eta^{12}+\frac{18}{25}\eta^{10}$$

$$-\frac{27}{2}\eta^8-12\eta^6+\frac{657}{10}\eta^4-\frac{342}{5}\eta^2\Big) \tag{35}$$

三次项及更高次项的解原则上也不难求得。为了求得 u_3，v_3 引入流函数 ϕ_3 使

$$\left.\begin{aligned} u_3 &= \frac{1}{\eta}\frac{\partial\phi_3}{\partial\theta} + \phi_2\sin\theta + \eta\phi_1\sin\theta\cos\theta \\ v_3 &= -\frac{1}{\eta}\frac{\partial\phi_3}{\partial\eta} + \frac{1}{\eta}\phi_2\cos\theta + \phi_1\cos^2\theta \end{aligned}\right\} \tag{36}$$

然后不难从纳维-斯托克斯方程中得到 ϕ_3 的如下方程

$$L\phi_3 = f(\eta,\theta) \tag{37}$$

其中 $f(\eta,\theta)$ 是 $\eta^m\cos n\theta$，$\eta^k\sin l\theta$ 型函数的线性组合。相应方程(37)的齐次方程的通解是

$$\begin{aligned} \phi_g &= c_{10} + c_{20}\ln\eta + c_{30}\eta^2\ln\eta + c_{40}\eta^2 \\ &+ \left(c_{11}\frac{1}{\eta} + c_{21}\eta\ln\eta + c_{31}\eta + c_{41}\eta^3\right)(A_1\cos\theta + B_1\sin\theta) \\ &+ \sum_{n=2}^{\infty}(c_{1n}\eta^{-n} + c_{2n}\eta^{-n+2} + c_{3n}\eta^n + c_{4n}\eta^{n+2})(A_n\cos n\theta + B_n\sin n\theta) \end{aligned} \tag{38}$$

故方程(37)的满足边界条件的解原则上不难求得。更高次项的解也可同样求得。但具体计算过于繁复，本文只计算到二次项。

3. 曲率和挠率对流动的影响

我们引入一个正交归一的坐标系，该坐标系的基矢为

$$\langle \boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*\rangle \equiv \left\langle \frac{\boldsymbol{e}_1}{|\boldsymbol{e}_1|}, \frac{\boldsymbol{e}_2}{|\boldsymbol{e}_2|}, \frac{\boldsymbol{e}_1\times\boldsymbol{e}_2}{|\boldsymbol{e}_1\times\boldsymbol{e}_2|}\right\rangle,$$

速度矢量 $\boldsymbol{V}$ 在该基矢中的分解为

$$\boldsymbol{V} = U(u^*\boldsymbol{e}_1^* + v^*\boldsymbol{e}_2^* + w^*\boldsymbol{e}_3^*) \tag{39}$$

如果只计算到二阶摄动解，则有

$$u^* = u_1\varepsilon + u_2\varepsilon^2 \tag{40}$$

$$v^* = (v_1\eta + w_0\eta\alpha)\varepsilon + (v_2\eta + w_0\eta\alpha)\varepsilon^2 \tag{41}$$

$$w^* = w_0 + (w_1 - w_0\eta\cos\theta)\varepsilon + (w_2 - w_1\eta\cos\theta)\varepsilon^2 \tag{42}$$

弯管内的主流流量可按下式计算

$$Q = 2\pi U\int_0^{2\pi}\int_0^a w^* r\,dr\,d\theta \tag{43}$$

由于 α 仅在 w_2 内出现，并总是出现在周期项，故在二级近似的范围内挠率对流量没有影响。ε 一次项的系数都含有周期函数，所以在一级近似的范围内，曲率对流量没有影响。积分(43)式得

$$\frac{Q}{\left(\frac{1}{2}\pi U a^2\right)} = 1 - \frac{\varepsilon^2}{48}\left[\frac{1541}{67200}\left(\frac{R}{6}\right)^2 + \frac{11}{10}\left(\frac{R}{6}\right)^2 - 1\right] \tag{44}$$

为了显示两次流动，我们再引进一个实际意义上的流函数。在螺旋坐标系中

$$\phi = \int_\eta^1 v\,d\eta \tag{45}$$

在正交归一的工程坐标系中

$$\psi^* = \int_\eta^1 v^* d\eta \tag{46}$$

由(45)、(46)通过不算复杂的代数运算可以求得 ψ，ψ^*；这二个函数的等值线即是二个不同坐标系下二次流动的流线．在一次近似的范围内，在工程坐标系中挠率对二次流没有影响；而在螺旋坐标系中对二次涡则有明显的压缩效应．图 1 展示了在这二个坐标系中，相应于不同的雷诺数、曲率、挠率的二阶摄动解的二次流动流线．图中显示的二次涡

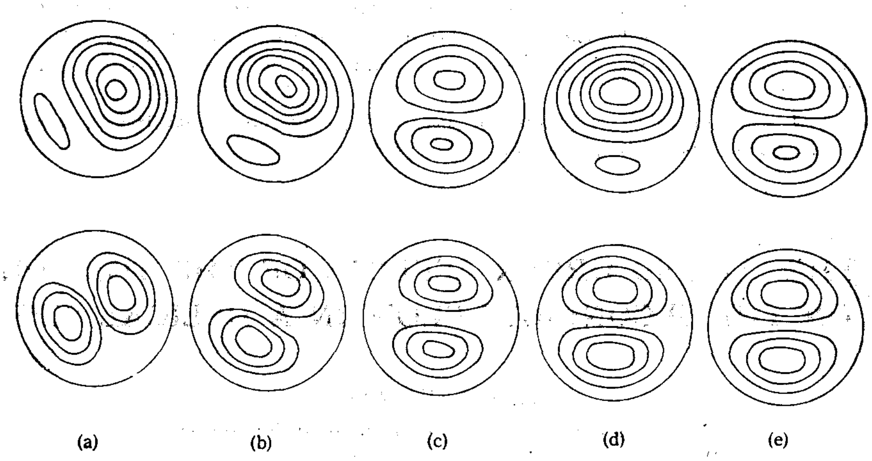

图 1　二次流动(第一排是 ψ 的等值线，第二排是 ψ^* 的等值线，$R=20$)
(a) $\alpha=1, \varepsilon=0.2$　(b) $\alpha=0.5, \varepsilon=0.1$　(c) $\alpha=0.1, \varepsilon=0.1$　(d) $\alpha=0.5$, $\varepsilon=0.001$
(e) $\alpha=0.05$　$\varepsilon=0.001$

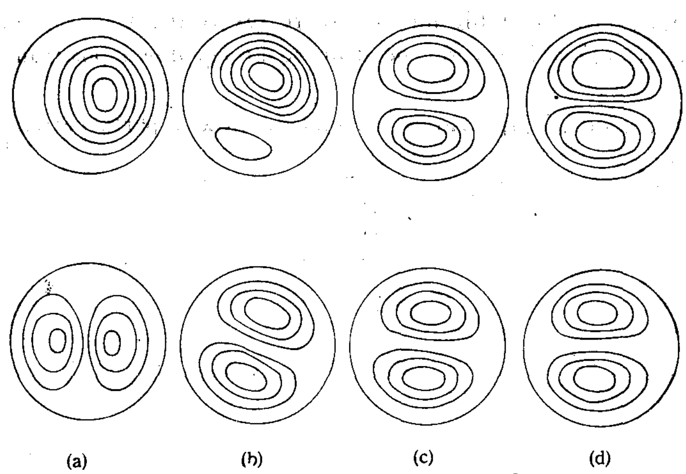

图 2　二次涡的旋转(第一排是 ψ 的等值线，第二排是 ψ^* 的等值线 $\alpha=0.1, \varepsilon=0.1$)
(a) $R=0.01$　(b) $R=5$　(c) $R=20$　(d) $R=50$

的扭曲和偏转是摄动解的二阶效应．图 2 则展现了当雷诺数由零渐次增大时，二次涡慢

慢旋转的有趣图象. 鉴于收敛性的要求,摄动法仅适用于 $R < \frac{17}{\sqrt{\varepsilon}}$ 的情况[4], 对于更大的雷诺数,数值计算就不可避免了.

周红宇同志为本文绘制了插图,特此致谢.

参 考 文 献

[1] Berger, S. A. and Talbot, L., Flow in curved pipes. *Ann. Rev. Fluid Mech.*, 15(1983), 46—512

[2] Wang, C. Y., On the low-Reynolds-number flow in a helical pipe, *J. Fluid Mech.*, 108(1981), 185—194

[3] Murata, S, Mijake, Y., Inaba, T. and Ogawa, H. Laminar flow in a helically coiled pipe, Bull. *JSME*, 24(1981), 355—362

[4] Larrain, J. and Bonilla, C. F., Theoretical analysis of pressure drop in the laminar flow of fluid in a coiled pipe, *Trans. Soc. Rheol.*, 14(1970), 135—147

THE SECOND ORDER EFFECTS OF THE TORSION ON THE SECONDARY FLOW IN A HELICAL PIPE

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Abstract A steady incompressible viscous flow in a helical pipe with low Reynolds number ($R < 17/\sqrt{\varepsilon}$) is investigated by the perturbation method and the usage of helical coordinate system. A second order solution is fully presented. Results show that the torsion of the pipe has turning and deforming effects on the secondary vortices. As Reynolds number increases from, two vortices of the secondary flow turn around a left-right position to an up-down position.

Key words Helical pipe flows, secondary flows, perturbation methods