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三元非定常流的面迹法

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提要 本文对一般性的三维非定常流动，通常引入面迹的概念，将三维非定常流动(或定常流)转化为面迹上的二维非定常流动或三维定常流动问题。 该理论为解三维非定常流或跨音流提供了一种可能的新途径。对叶轮机内较复杂的流动问题，可通过该理论来简化研究模型，使理论和实验研究得以简化。

关键词 非定常流，跨音流，叶轮机

1. 引言

对三维非定常流问题进行研究，不论是理论研究还是实验研究都是一项复杂和费用昂贵的工作。若能将问题简化为二维非定常流问题，则其工作量和研究成本将大大减少。对叶轮机内定常流动，吴仲华先生的两类相对流面理论[1]成功地将三维定常流问题转化成流面上的二维定常流问题。对三维非定常流，流面可能是随时间变化的，所以流面理论不再适用。本文旨在建立三维非定常流的相应理论。

2. 面迹的概念和性质

首先提出物质面的概念，物质面是指流场中流体微团组成的曲面。面迹是指随时间的推移，流场中某初始指定的物质面随其上流体微团运动而运动的变化历程。对任一瞬时，面迹对应一个由初始物质面上的流体微团组成的，空间位置和形状都发生了变化的空间曲面，且可以证明：初始指定的连续的物质面在流动中将保持连续。面迹在直角座标系中可写为 $F(x,y,z,t)=0$，若将其写成各种显式形式，由此将导出不同形式的变换后方程，据此定义各种类型的面迹。定义 $t=t(x,y,z)$ 形式的面迹方程为 S 型面迹；定义 $x=x(y,z,t)$ 为 $U1$ 型面迹，$y=y(x,z,t)$ 为 $U2$ 型面迹，$z=z(x,y,t)$ 为 $U3$ 型面迹，$U1$、$U2$、$U3$ 统称为 U 型面迹。

由于圆柱坐标系中，面迹的概念、性质、及其上方程的推导与直角坐标系中完全类似，故以后以直角坐标系为主进行推导。

流体微团沿迹线

$$\frac{dx}{dt}=u \quad \frac{dy}{dt}=v \quad \frac{dz}{dt}=w \tag{1}$$

对 S 面迹，t 时刻，$t=t(x,y,z)$ 为一流体微团面，这个面上所有流体微团在 $t+dt$

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时刻仍组成一个曲面 $t+dt=t(x+dx,\ y+dy,\ z+dz)$.

$$dt=t(x+dx,y+dy,z+dz)-t(x,y,z)$$

对等式右边作泰勒展开，并取其一阶项得

$$dt=\frac{\bar{\partial} t}{\partial x}dx+\frac{\bar{\partial} t}{\partial y}dy+\frac{\bar{\partial} t}{\partial z}dz$$

此处及以后的 $\frac{\bar{\partial}}{\partial}$ 均指对面迹求偏导. 上式两边同除以 dt 并由(1)式得

$$1=u\frac{\bar{\partial} t}{\partial x}+v\frac{\bar{\partial} t}{\partial y}+w\frac{\bar{\partial} t}{\partial z} \tag{2}$$

对 U 型面迹，与 S 面迹类似. 对 $U1$ 面迹 $x=x(y,z,t)$ 有

$$u=\frac{\bar{\partial} x}{\partial t}+v\frac{\bar{\partial} x}{\partial y}+w\frac{\bar{\partial} x}{\partial z} \tag{3}$$

对 $U2$ 面迹 $y=y(x,z,t)$，有

$$v=\frac{\bar{\partial} y}{\partial t}+u\frac{\bar{\partial} y}{\partial x}+w\frac{\bar{\partial} y}{\partial z} \tag{4}$$

对 $U3$ 面迹 $z=z(x,y,t)$，有

$$w=\frac{\bar{\partial} z}{\partial t}+u\frac{\bar{\partial} z}{\partial x}+v\frac{\bar{\partial} z}{\partial y} \tag{5}$$

任一参数 q 沿面迹的偏导数和随流导数. 对 S 型面迹，$t=t(x,\ y,\ z)$，任一参数 $q=q(x,y,z,t)=q(x,y,z,t(x,y,z))$，故

$$\frac{\bar{\partial} q}{\partial x}=\frac{\partial q}{\partial x}+\frac{\partial q}{\partial t}\cdot\frac{\bar{\partial} t}{\partial x} \tag{6}$$

$$\frac{\bar{\partial} q}{\partial y}=\frac{\partial q}{\partial y}+\frac{\partial q}{\partial t}\frac{\bar{\partial} t}{\partial y} \tag{7}$$

$$\frac{\bar{\partial} q}{\partial z}=\frac{\partial q}{\partial z}+\frac{\partial q}{\partial t}\cdot\frac{\bar{\partial} t}{\partial z} \tag{8}$$

随流导数
$$\frac{Dq}{Dt}=u\frac{\bar{\partial} q}{\partial x}+v\frac{\bar{\partial} q}{\partial y}+w\frac{\bar{\partial} q}{\partial z} \tag{9}$$

U 面迹以 $U1$ 面迹为例，其余类推. $U1$ 面迹上

$$q=q(x,y,z,t)=q(x(y,z,t),y,z,t).$$

$$\frac{\bar{\partial} q}{\partial t}=\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}\frac{\bar{\partial} x}{\partial t} \tag{10}$$

$$\frac{\bar{\partial} q}{\partial y}=\frac{\partial q}{\partial y}+\frac{\partial q}{\partial x}\frac{\bar{\partial} x}{\partial y} \tag{11}$$

$$\frac{\bar{\partial} q}{\partial z}=\frac{\partial q}{\partial z}+\frac{\partial q}{\partial x}\frac{\bar{\partial} x}{\partial z} \tag{12}$$

随流导数

$$\frac{Dq}{Dt}=\frac{\bar{\partial} q}{\partial t}+v\frac{\bar{\partial} q}{\partial y}+w\frac{\bar{\partial} q}{\partial z} \tag{13}$$

3. 面迹上的方程

1) 三元非定常流基本方程

连续方程

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{14}$$

动量方程

$$\frac{Du}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+f_x \tag{15}$$

$$\frac{Dv}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+f_y \tag{16}$$

$$\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+f_z \tag{17}$$

能量方程

$$\frac{Di^*}{Dt}=\frac{Dq}{Dt}+\frac{1}{\rho}\frac{\partial p}{\partial t}+\boldsymbol{w}\cdot\boldsymbol{f}+\phi/\rho \tag{18}$$

2) S 型面迹上的基本方程

在面迹上基本方程推导时，所有的粘性项都当作已知函数考虑，故其形式不变。

连续方程，由(14)及(6)、(7)、(8)得

$$\frac{\bar{\partial}(\rho u)}{\partial x}+\frac{\bar{\partial}(\rho v)}{\partial y}+\frac{\bar{\partial}(\rho w)}{\partial z}=-\rho\left(\frac{\partial u}{\partial t}\frac{\bar{\partial}t}{\partial x}+\frac{\partial v}{\partial t}\frac{\bar{\partial}t}{\partial y}+\frac{\partial w}{\partial t}\frac{\bar{\partial}t}{\partial z}\right)$$

令

$$\frac{D\ln b_t}{Dt}=-\left(\frac{\partial u}{\partial t}\frac{\bar{\partial}t}{\partial x}+\frac{\partial v}{\partial t}\frac{\bar{\partial}t}{\partial y}+\frac{\partial w}{\partial t}\frac{\bar{\partial}t}{\partial z}\right)$$

得

$$\frac{\bar{\partial}(b_t\rho u)}{\partial x}+\frac{\bar{\partial}(b_t\rho v)}{\partial y}+\frac{\bar{\partial}(b_t\rho w)}{\partial z}=0$$

动量方程，对 x 方向，将(6)、(9)代入(15)得

$$u\frac{\bar{\partial}u}{\partial x}+v\frac{\bar{\partial}u}{\partial y}+w\frac{\bar{\partial}u}{\partial z}=-\frac{1}{\rho}\frac{\bar{\partial}p}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial t}\frac{\bar{\partial}t}{\partial x}+f_x=-\frac{1}{\rho}\frac{\bar{\partial}p}{\partial x}+F_{tx}+f_x$$

类似于对 x 方向，对 y、z 方向有

$$u\frac{\bar{\partial}v}{\partial x}+v\frac{\bar{\partial}v}{\partial y}+w\frac{\bar{\partial}v}{\partial z}=-\frac{1}{\rho}\frac{\bar{\partial}p}{\partial y}+F_{ty}+f_y$$

$$u\frac{\bar{\partial}w}{\partial x}+v\frac{\bar{\partial}w}{\partial y}+w\frac{\bar{\partial}w}{\partial z}=-\frac{1}{\rho}\frac{\bar{\partial}p}{\partial z}+F_{tz}+f_z$$

此处

$$F_{tx}=\frac{1}{\rho}\frac{\partial p}{\partial t}\frac{\bar{\partial}t}{\partial x},\quad F_{ty}=\frac{1}{\rho}\frac{\partial p}{\partial t}\frac{\bar{\partial}t}{\partial y},\quad F_{tz}=\frac{1}{\rho}\frac{\partial p}{\partial t}\frac{\bar{\partial}t}{\partial z}.$$

对定常流，$\frac{\partial p}{\partial t}=0$，则 $F_{tx}=F_{ty}=F_{tz}=0$。一般可令 $N_{tt}=\frac{1}{\rho}\frac{\partial p}{\partial t}\neq 0$，则

$$\frac{\bar{\partial}t}{\partial x}=\frac{F_{tx}}{N_{tt}},\quad \frac{\bar{\partial}t}{\partial y}=\frac{F_{ty}}{N_{tt}},\quad \frac{\bar{\partial}t}{\partial z}=\frac{F_{tz}}{N_{tt}}.$$

将(2)两边同乘以 $\frac{1}{\rho}\frac{\partial p}{\partial t}$，得 $N_{tt}=uF_{tx}+vF_{ty}+wF_{tz}$。

能量方程，将 $N_{tt} = \frac{1}{\rho}\frac{\partial p}{\partial t}$ 代入(18)式，得：

$$\frac{Di^*}{Dt} = \frac{Dq}{Dt} + N_{tt} + \phi/\rho + \boldsymbol{w}\cdot\boldsymbol{f}$$

3）U 面迹上的方程

对 U 面迹上的方程推导以 $U1$ 面迹为例，对 $U2$、$U3$ 面迹完全类似。

连续方程　将式(10)、(11)、(12)代入(14)，连续方程化为

$$\frac{\bar{\partial}(b_1\rho)}{\partial t} + \frac{\bar{\partial}(b_1\rho v)}{\partial y} + \frac{\bar{\partial}(b_1\rho w)}{\partial z} = 0$$

其中
$$\frac{D\ln b_1}{Dt} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\bar{\partial} x}{\partial y} - \frac{\partial w}{\partial x}\frac{\bar{\partial} x}{\partial z}.$$

动量方程　将(13)式代入(15)式，对 x 方向

$$\frac{\bar{\partial} u}{\partial t} + v\frac{\bar{\partial} u}{\partial y} + w\frac{\bar{\partial} u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x = F_{1x} + f_x$$

类似地，对 y、z 方向有

$$\frac{\bar{\partial} v}{\partial t} + v\frac{\bar{\partial} v}{\partial y} + w\frac{\bar{\partial} v}{\partial z} = -\frac{1}{\rho}\frac{\bar{\partial} p}{\partial y} + F_{1y} + f_y$$

$$\frac{\bar{\partial} w}{\partial t} + v\frac{\bar{\partial} w}{\partial y} + w\frac{\bar{\partial} w}{\partial z} = -\frac{1}{\rho}\frac{\bar{\partial} p}{\partial z} + F_{1z} + f_z$$

此处 $F_{1x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$，$F_{1y} = \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\bar{\partial} x}{\partial y}$，$F_{1z} = \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\bar{\partial} x}{\partial z}$。

令
$$N_{1t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\bar{\partial} x}{\partial t},$$

将(3)式两边同乘以 $-\frac{1}{\rho}\frac{\partial p}{\partial x}$，得

$$uF_{1x} + vF_{1y} + wF_{1z} = N_{1t}$$

能量方程　由 $\frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{\bar{\partial} p}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\bar{\partial} x}{\partial t} = \frac{1}{\rho}\frac{\bar{\partial} p}{\partial t} + N_{1t}$，能量方程(18)式变为

$$\frac{Di^*}{Dt} = \frac{Dq}{Dt} + \frac{1}{\rho}\frac{\bar{\partial} p}{\partial t} + N_{1t} + \phi/\rho$$

4. 讨论

从上述面迹上的方程可看出：S 型面迹上的方程是三元定常流形式；U 型面迹上的方程是二元非定常流形式，可以证明它们是双曲型、弱守恒方程组。对理想流，可以看出，S 和 U 面迹上的方程组都是封闭的。

S 型面迹上的方程的求解域是在一定的时间历程中，某初始指定的物质面其运动轨迹所形成的三维空间。U 型面迹上方程的求解域是面迹在任一瞬时对应的曲面。实际解题时，面迹类型的选择取决于解题的方便性；若已知某簇面迹，则用该簇面迹上的方程配以适当的边界条件即可解出流场。如对理想流，固壁 $B(x, y, z, t) = 0$，有：

对 S 面迹 $$u\frac{\bar{\partial} B}{\partial x}+v\frac{\bar{\partial} B}{\partial y}+w\frac{\bar{\partial} B}{\partial z}=0$$

对 $U1$ 面迹 $$\frac{\bar{\partial} B}{\partial t}+v\frac{\bar{\partial} B}{\partial y}+w\frac{\bar{\partial} B}{\partial z}=0$$

而一般在未知或不能确切知道面迹的情况下,需要面迹簇之间的迭代.

对定常流,若采用 S 型面迹,则其上述方程组中,$F_{tx}=F_{ty}=F_{tz}=0$, $N_{tt}=0$; 以 x 向偏导数为例: $\frac{\bar{\partial} q}{\partial x}=\frac{\partial q}{\partial x}+\frac{\partial q}{\partial t}\frac{\bar{\partial} t}{\partial x}$, 而 $\frac{\partial q}{\partial t}=0$, 所以 $\frac{\bar{\partial} q}{\partial x}=\frac{\partial q}{\partial x}$. 即 S 型方程在定常流下与未变换的方程完全一致. 若采用 U 型面迹,当初始物质面不是流面时,则通过面迹,三维定常流方程变为二维非定常流的形式. 比常用的时间相关法少一维,故它也是解决跨音流的一个可能途径. 当初始物质面正好取在流面上,则面迹退化为流面,U 面迹上的方程变为与流面上方程一致的二维定常流方程组. 可见,面迹理论是流面理论的推广.

对非定常流,若采用 S 型面迹,则流动方程在面迹上变为三维定常流方程的形式. 若采用 U 型面迹,则面迹上的方程为二维非定常流的形式,问题就降了一维. 特别当面迹不随时间变化时(或变化小到可忽略),则面迹也是一个流面,其上方程为二维非定常流方程. 由此可推知: 如果流面是随时间变化的,则其上不成立上述推导的方程,亦既不能通过流面来使三元非定常流降维.

对叶轮机内复杂的非定常流动问题,由于叶轮机的结构特征. 有可能较方便地知道或近似假定出面迹,从而可简化对其理论和实验研究.

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THE METHOD OF TRACES OF SUBSTANTIAL SURFACES FOR THE SOLUTION OF THREE-DIMENSIONAL UNSTEADY FLOW

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Abstract Through the concept of traces of substantial surface (T. S. for short), problems of three-dimensional unsteady (or steady) flow are transformed into those of two-dimensional unsteady or three-dimensional steady flow. This theory provides a new approach to the solution of three-dimensional unsteady flow or steady transonic flow. According to this theory, flow models, which makes theoretical research easier and experimental research more convenient and economic, could be found to simplify complex flow problems in turbomachine.

Key words unsteady flow, transonic flow, turbomachinery