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变质量高阶非线性非完整系统的广义 Volterra 方程

乔 永 芬
(东 北 农 学 院)

提要　本文由变质量力学系统的万有 D'Alembert 原理导出变质量高阶非线性非完整系统的广义 Volterra 方程.

关键词　变质量,高阶非完整系统,广义 Volterra 方程

1. 引言

1898年意大利著名数学家 V. Volterra 利用他提出的"运动学特性"的新概念,建立了力学系统的一类运动方程[1]. 这些方程称为 Volterra 方程. 它适用于线性齐次、稳定的非完整系统. 1985年梅凤翔将 Volterra 方程推广到非线性非完整系统[2].

本文由变质量力学系统的万有 D'Alembert 原理导出变质量高阶非线性非完整系统的广义 Volterra 方程.

2. 变质量高阶非线性非完整系统的 Volterra 方程

设有N个质点组成的变质量力学系统，相对于惯性坐标系运动，其位形由直角坐标 $x_1, y_1, z_1, \cdots, x_N, y_N, z_N$ 确定,点的质量为 $M_1(t), M_2(t), \cdots, M_N(t)$. 令 $\xi_1 = x_1$, $\xi_2 = y_1$, $\xi_3 = z_1$, $\cdots$, $\xi_{3N-2} = x_N$, $\xi_{3N-1} = y_N$, $\xi_{3N} = z_N$; $m_1 = m_2 = m_3 = M_1(t)$, $\cdots$, $m_{3N-2} = m_{3N-1} = m_{3N} = M_N(t)$;作用于质点上的主动力为 $X_1, X_2, X_3, \cdots, X_{3N-2}, X_{3N-1}, X_{3N}$; 反推力为 $X_1^R, X_2^R, X_3^R, \cdots, X_{3N-2}^R, X_{3N-1}^R, X_{3N}^R$.

设变质量系统受 l 个理想完整约束

$$F_\alpha(\xi_i, t) = 0 \tag{2.1}$$

$$(\alpha = 1, 2, \cdots, l;\ i = 1, 2, \cdots, 3N)$$

以及 g 个 m 阶非线性非完整约束

$$f_\beta(\xi_i, \dot{\xi}_i, \cdots, \overset{(m-1)}{\xi_i}, \overset{(m)}{\xi_i}, t) = 0 \tag{2.2}$$

$$(\beta = 1, 2, \cdots, g;\ i = 1, 2, \cdots, 3N;\ m = 1, 2, \cdots)$$

将(2.1)对时间 t 求 m 阶导数，有

$$\sum_{i=1}^{3N} \frac{\partial F_\alpha}{\partial \xi_i} \overset{(m)}{\xi_i} + G = 0 \tag{2.3}$$

其中G为不含 $\overset{(m)}{\xi_i}$ 之项.

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引入 $\varepsilon = n - g(n = 3N - l)$ 个运动学特性 $\overset{(m-1)}{P_\sigma}$，使点的$m$阶速度为

$$\overset{(m)}{\xi_i} = \overset{(m)}{\xi_i}(\xi_j, \dot{\xi}_j, \cdots, \overset{(m-1)}{\xi_j}, \overset{(m-1)}{P_\sigma}, t) \tag{2.4}$$

$$(i, j = 1, 2, \cdots, 3N; \sigma = 1, 2, \cdots \varepsilon; m = 1, 2, \cdots)$$

而(2.2)和(2.3)成为恒等式，即

$$f_\beta(\xi_i, \dot{\xi}_i, \cdots, \overset{(m-1)}{\xi_i}, \overset{(m)}{\xi_i}(\xi_j, \dot{\xi}_j, \cdots, \overset{(m-1)}{\xi_j}, \overset{(m-1)}{P_\sigma}, t), t) \equiv 0 \tag{2.5}$$

$$\sum_{i=1}^{3N} \frac{\partial F_\alpha}{\partial \xi_i} \overset{(m)}{\xi_i}(\xi_j, \dot{\xi}_j, \cdots, \overset{(m-1)}{\xi_j}, \overset{(m-1)}{P_\sigma}, t) + G \equiv 0 \tag{2.6}$$

变质量系统万有 D'Alembert-Lagrange 原理可以写成如下形式:

$$\sum_{i=1}^{3N} (-m_i \ddot{\xi}_i + X_i + X_i^R) \delta \overset{(m)}{\xi_i} = 0 \tag{2.7}$$

$$(\delta \xi_i = \delta \dot{\xi}_i = \cdots = \delta \overset{(m-1)}{\xi_i} = 0; \delta t = 0; \delta \overset{(m)}{\xi_i} \neq 0; m = 1, 2, \cdots)$$

我们研究理想约束情形，此时原理(2.7)中的 X_i 便是作用于质点上的主动力分量.

由(2.4)，广义虚位移满足条件

$$\delta \overset{(m)}{\xi_i} = \sum_{\sigma=1}^{\varepsilon} \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} \delta \overset{(m-1)}{P_\sigma} \tag{2.8}$$

其中 $\delta \overset{(m-1)}{P_\sigma}$ 为运动学特性 $\overset{(m-1)}{P_\sigma}$ 的变分. 将(2.8)代入原理(2.7)，由于 $\delta \overset{(m-1)}{P_\sigma}$ 彼此独立，得到 Maggi 型方程

$$-\sum_{i=1}^{3N} m_i \ddot{\xi}_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} + \sum_{i=1}^{3N} X_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} + \sum_{i=1}^{3N} X_i^R \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} = 0 \tag{2.9}$$

$$(\sigma = 1, 2, \cdots, \varepsilon)$$

现在变换方程 (2.9).

设系统中各质点的质量为 $m_i = m_i(t)$.

用直角坐标表示的系统动能

$$T_0 = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{\xi}_i^2$$

$$\dot{T}_0 = \sum_{i=1}^{3N} m_i \dot{\xi}_i \ddot{\xi}_i + \frac{1}{2} \sum_{i=1}^{3N} \dot{m}_i \dot{\xi}_i^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overset{(m)}{T_0} = \sum_{i=1}^{3N} m_i \dot{\xi}_i \overset{(m+1)}{\xi_i} + m \sum_{i=1}^{3N} m_i \ddot{\xi}_i \overset{(m)}{\xi_i} + m \sum_{i=1}^{3N} \dot{m}_i \dot{\xi}_i \overset{(m)}{\xi_i} + \cdots +$$

$$+ \sum_{i=1}^{3N} \overset{(m)}{m_i} \frac{1}{2} \dot{\xi}_i^2$$

由上可得

$$\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}} = \frac{\partial T_0}{\partial \dot{\xi}_i} \tag{2.10}$$

令 $\overset{(m-1)}{T}$ 为 $\overset{(m-1)}{T_0}$ 中借助于(2.4)消去 $\overset{(m)}{\xi_i}$ 而得表达式,有

$$\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} = \sum_{i=1}^{3N} \frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}} \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} = \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} \tag{2.11}$$

于是

$$\frac{d}{dt} \frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} = \sum_{i=1}^{3N} m_i \ddot{\xi}_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} + \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{d}{dt}\left(\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}\right) + \sum_{i=1}^{3N} \dot{m}_i \dot{\xi}_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}$$

将上式代入方程(2-9),则得

$$\frac{d}{dt} \frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{d}{dt}\left(\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}\right) = \tilde{E}_\sigma + \tilde{\Phi}'_\sigma \tag{2.12}$$

$$(\sigma = 1, 2, \cdots, \varepsilon)$$

其中

$$\tilde{E}_\sigma = \sum_{i=1}^{3N} X_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}$$

$$\tilde{\Phi}'_\sigma = \sum_{i=1}^{3N} (X_i^R + \dot{m}_i \dot{\xi}_i) \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}$$

我们称方程(2.12)为变质量高阶非线性非完整系统第一形式的广义 Volterra 方程。

如果 $m = 1$，则(2.12)式成为

$$\frac{d}{dt} \frac{\partial T}{\partial P_\sigma} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{d}{dt}\left(\frac{\partial \dot{\xi}_i}{\partial P_\sigma}\right) = \tilde{E}_\sigma + \tilde{\Phi}'_\sigma \tag{2.13}$$

$$(\sigma = 1, 2, \cdots, \varepsilon)$$

其中

$$\tilde{E}_\sigma = \sum_{i=1}^{3N} X_i \frac{\partial \dot{\xi}_i}{\partial P_\sigma} \qquad \tilde{\Phi}'_\sigma = \sum_{i=1}^{3N} (X_i^R + \dot{m}_i \dot{\xi}_i) \frac{\partial \dot{\xi}_i}{\partial P_\sigma}$$

方程(2.13)就是变质量一阶非完整系统第一形式的广义 Volterra 方程[3]。

如果 $m = 1$，且 $m_i(t) =$ 常量,此时方程(2.13)可化为

$$\frac{d}{dt} \frac{\partial T}{\partial P_\sigma} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{d}{dt}\left(\frac{\partial \dot{\xi}_i}{\partial P_\sigma}\right) = \tilde{E}_\sigma \ (\sigma = 1, 2, \cdots, \varepsilon) \tag{2.14}$$

方程(2.14)与文献[2]的结果相同。

现在引入对准坐标 $\overset{(m-1)}{\pi_\sigma}$ 的偏导数记号

$$\frac{\partial}{\partial \overset{(m-1)}{\pi_\sigma}} \triangleq \sum_{i=1}^{3N} \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} \frac{\partial}{\partial \overset{(m-1)}{\xi_i}} \tag{2.15}$$

于是，有

$$\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\pi_\sigma}} = \sum_{i=1}^{3N} \frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\xi_i}} \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} \tag{2.16}$$

由于

$$\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\xi_i}} = \sum_{j=1}^{3N} \frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_j}} \frac{\partial \overset{(m)}{\xi_j}}{\partial \overset{(m-1)}{\xi_i}} + \frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m-1)}{\xi_i}} \tag{2.17}$$

将(2.17)代入(2.16)，并注意(2.15)，得

$$\begin{aligned}\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\pi_\sigma}} &= \sum_{i=1}^{3N}\left[\sum_{j=1}^{3N} \frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_j}} \frac{\partial \overset{(m)}{\xi_j}}{\partial \overset{(m-1)}{\xi_i}} + \frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m-1)}{\xi_i}}\right] \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}} \\ &= \sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{\pi_\sigma}} + (m-1)\frac{d}{dt}\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} \\ &\quad - (m-1)\sum_{i=1}^{3N} m_i \dot{\xi}_i \frac{d}{dt}\left(\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}\right)\end{aligned} \tag{2.18}$$

再将(2.18)代入(2.12)，经过整理后，有

$$\begin{aligned} m\frac{d}{dt}\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} - \frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\pi_\sigma}} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \left[m \frac{d}{dt}\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{P_\sigma}}\right. \\ \left. - \frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{\pi_\sigma}}\right] = \tilde{E}_\sigma + \tilde{\Phi}'_\sigma \quad (\sigma = 1, 2, \cdots, \varepsilon)\end{aligned} \tag{2.19}$$

我们称方程(2.19)为变质量高阶非线性非完整系统第二形式的广义 Volterra 方程.

如果 $m = 1$，则方程(2.19)成为

$$\frac{d}{dt}\frac{\partial T}{\partial P_\sigma} - \frac{\partial T}{\partial \pi_\sigma} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \left(\frac{d}{dt}\frac{\partial \dot{\xi}_i}{\partial P_\sigma} - \frac{\partial \dot{\xi}_i}{\partial \pi_\sigma}\right) = \tilde{E}_\sigma + \tilde{\Phi}'_\sigma \tag{2.20}$$

$$(\sigma = 1, 2, \cdots, \varepsilon)$$

其中

$$\tilde{E}_\sigma = \sum_{i=1}^{3N} X_i \frac{\partial \dot{\xi}_i}{\partial P_\sigma} \qquad \tilde{\Phi}'_\sigma = \sum_{i=1}^{3N} (X_i^R + \dot{m}_i \dot{\xi}_i) \frac{\partial \dot{\xi}_i}{\partial P_\sigma}$$

方程(2.20)是变质量一阶非完整系统第二形式的广义 Volterra 方程.

如果 $m = 1$，且 $m_i(t) =$ 常量，此时方程(2.20)可化为

$$\frac{d}{dt}\frac{\partial T}{\partial P_\sigma} - \frac{\partial T}{\partial \pi_\sigma} - \sum_{i=1}^{3N} m_i \dot{\xi}_i \left(\frac{d}{dt}\frac{\partial \dot{\xi}_i}{\partial P_\sigma} - \frac{\partial \dot{\xi}_i}{\partial \pi_\sigma}\right) = \tilde{E}_\sigma \tag{2.21}$$

方程(2.21)与文献[2]的结果相同.

3. 变质量高阶非线性非完整系统广义坐标形式下的广义 Volterra 方程

设变质量力学系统中各点的质量 $m_i = m_i(t)$，系统受理想约束为(2.1)、(2.2).

设点的直角坐标 ξ_i 用 $n = 3N - l$ 个广义坐标 q_s 及时间 t 表出:

$$\xi_i=\xi_i(q_s,t)\quad(s=1,2,\cdots,n;\ i=1,2,\cdots,3N) \tag{3.1}$$

这样,完整约束可自动地得到满足。

$$\dot{\xi}_i=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\dot{q}_s+\frac{\partial\xi_i}{\partial t}$$

$$\ddot{\xi}_i=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\ddot{q}_s+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\xi_i}{\partial q_s\partial q_k}\dot{q}_s\dot{q}_k+2\sum_{s=1}^{n}\frac{\partial^2\xi_i}{\partial q_s\partial t}\dot{q}_s+\frac{\partial^2\xi_i}{\partial t^2}$$

……

$$\overset{(m)}{\xi}_i=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\overset{(m)}{q}_s+H\,(\text{不含}\ \overset{(m)}{q}_s\ \text{之项}) \tag{3.2}$$

由上可见

$$\frac{\partial\overset{(m)}{\xi}_i}{\partial\overset{(m)}{q}_s}=\frac{\partial\xi_i}{\partial q_s} \tag{3.3}$$

取运动学特性 $\overset{(m-1)}{P}_\sigma$ ，使

$$\overset{(m)}{q}_s=\overset{(m)}{q}_s(q_k,\dot{\xi}_j,\cdots,\overset{(m-1)}{\xi}_j,\overset{(m-1)}{P}_\sigma,t) \tag{3.4}$$

$$(s,k=1,2,\cdots,n;\ \sigma=1,2,\cdots,\varepsilon;\ m=1,2,\cdots;\ j=1,2,\cdots,3N)$$

将(3.4)代入(3.2),并记作 $(\overset{(m)}{\xi}_i)$:

$$(\overset{(m)}{\xi}_i)=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\overset{(m)}{q}_s(q_k,\dot{\xi}_j,\cdots,\overset{(m-1)}{\xi}_j,\overset{(m-1)}{P}_\sigma,t)+H \tag{3.5}$$

现在变换方程(2.19)中的各项.

令 $\overset{(m-1)}{T}_1$ 为借助(3-2)式消去 $\overset{(m-1)}{T}_0$ 中的 $\overset{(m)}{\xi}_i$ 而得的表达式,即

$$\overset{(m-1)}{T}_1=\overset{(m-1)}{T}_0\Big|_{\overset{(m)}{\xi}_i}=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\overset{(m)}{q}_s+H \tag{3.6}$$

令 $\overset{(m-1)}{\widetilde{T}}$ 为用(3.4)式代入以后所得的 $\overset{(m-1)}{T}_1$,即

$$\overset{(m-1)}{\widetilde{T}}=\overset{(m-1)}{T}_1\Big|_{\overset{(m)}{q}_s=\overset{(m)}{q}_s(q_k,\dot{\xi}_j,\cdots,\overset{(m-1)}{\xi}_j,\overset{(m-1)}{P}_\sigma,t)} \tag{3.7}$$

我们有:

$$\frac{\partial\overset{(m-1)}{\widetilde{T}}}{\partial\overset{(m-1)}{P}_\sigma}=\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T}_1}{\partial\overset{(m)}{q}_s}\frac{\partial\overset{(m)}{q}_s}{\partial\overset{(m-1)}{P}_\sigma}=\sum_{s=1}^{n}\sum_{i=1}^{3N}\frac{\partial\overset{(m-1)}{T}_0}{\partial\overset{(m)}{\xi}_i}\frac{\partial\overset{(m)}{\xi}_i}{\partial\overset{(m)}{q}_s}\frac{\partial\overset{(m)}{q}_s}{\partial\overset{(m-1)}{P}_\sigma}$$

$$=\sum_{i=1}^{3N}\frac{\partial\overset{(m-1)}{T}_0}{\partial\overset{(m)}{\xi}_i}\frac{\partial(\overset{(m)}{\xi}_i)}{\partial\overset{(m-1)}{P}_\sigma}=\frac{\partial\overset{(m-1)}{T}}{\partial\overset{(m-1)}{P}_\sigma} \tag{3.8}$$

亦即

$$\frac{d}{dt}\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{P_\sigma}}=\frac{d}{dt}\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{P_\sigma}} \tag{3.9}$$

$$\begin{aligned}\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{\pi_\sigma}}&=\sum_{s=1}^{n}\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}-\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m-1)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q_k}}\frac{\partial \overset{(m)}{q_k}}{\partial \overset{(m-1)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&-\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{\pi_\sigma}}+\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m-1)}{\xi_i}}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}\\&=\frac{\partial \overset{(m-1)}{T}}{\partial \overset{(m-1)}{\pi_\sigma}}\end{aligned} \tag{3.10}$$

由(3.5)式，我们有

$$\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}} \tag{3.11}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}&=\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&+\sum_{s=1}^{n}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\frac{d}{dt}\frac{\partial\xi_i}{\partial q_s}\end{aligned} \tag{3.12}$$

$$\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{\pi_\sigma}}=\sum_{s=1}^{n}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}} \tag{3.13}$$

于是

$$\begin{aligned}m\frac{d}{dt}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}-\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{\pi_\sigma}}&=m\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&+\sum_{s=1}^{n}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\left[m\frac{d}{dt}\frac{\partial\xi_i}{\partial q_s}-\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{q_s}}\right]\end{aligned} \tag{3.14}$$

又

$$m\frac{d}{dt}\frac{\partial\xi_i}{\partial q_s}-\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{q_s}}=-\sum_{k=1}^{n}\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m)}{q_k}}\frac{\partial \overset{(m)}{q_k}}{\partial \overset{(m-1)}{q_s}} \tag{3.15}$$

将(3.15)代入(3.14)，得

$$m\frac{d}{dt}\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}-\frac{\partial(\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{\pi_\sigma}}=m\sum_{s=1}^{n}\frac{\partial\xi_i}{\partial q_s}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}$$

$$-\sum_{s=1}^{n}\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}\sum_{k=1}^{n}\frac{\partial \overset{(m)}{\xi}_i}{\partial \overset{(m)}{q}_k}\frac{\partial \overset{(m)}{q}_k}{\partial \overset{(m-1)}{q}_s}$$

$$=\sum_{s=1}^{n}\frac{\partial \xi_i}{\partial q_s}\left(m\frac{d}{dt}\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}-\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{\pi}_\sigma}\right) \tag{3.16}$$

最后，将(3.9)，(3.10)和(3.16)代入到广义 Volterra 方程(2.19)，得到

$$m\frac{d}{dt}\frac{\partial \widetilde{\overset{(m-1)}{T}}}{\partial \overset{(m-1)}{P}_\sigma}-\frac{\partial \widetilde{\overset{(m-1)}{T}}}{\partial \overset{(m-1)}{\pi}_\sigma}-\sum_{i=1}^{3N} m_i\dot{\xi}_i\sum_{s=1}^{n}\frac{\partial \xi_i}{\partial q_s}\left(m\frac{d}{dt}\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}\right.$$

$$\left.-\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{\pi}_\sigma}\right)=\tilde{E}_\sigma+\tilde{\Phi}'_\sigma \tag{3.17}$$

由于

$$\sum_{i=1}^{3N} m\dot{\xi}_i\sum_{s=1}^{n}\frac{\partial \xi_i}{\partial q_s}=\sum_{s=1}^{n}\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi}_i}\frac{\partial \overset{(m)}{\xi}_i}{\partial \overset{(m)}{q}_s}=\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q}_s}$$

于是，(3.17)可化为

$$m\frac{d}{dt}\frac{\partial \widetilde{\overset{(m-1)}{T}}}{\partial \overset{(m-1)}{P}_\sigma}-\frac{\partial \widetilde{\overset{(m-1)}{T}}}{\partial \overset{(m-1)}{\pi}_\sigma}-\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q}_s}\left(m\frac{d}{dt}\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}\right.$$

$$\left.-\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{\pi}_\sigma}\right)=\tilde{E}_\sigma+\tilde{\Phi}'_\sigma \tag{3.18}$$

$$(\sigma=1,2,\cdots,\varepsilon)$$

其中

$$\tilde{E}_\sigma=\sum_{s=1}^{n}Q_s\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}\qquad Q_s=\sum_{i=1}^{3N}X_i\frac{\partial \xi_i}{\partial q_s}$$

$$\tilde{\Phi}'_\sigma=\sum_{s=1}^{n}\phi'_s\frac{\partial \overset{(m)}{q}_s}{\partial \overset{(m-1)}{P}_\sigma}\qquad \phi'_s=\sum_{i=1}^{3N}(X_i^R+\dot{m}_i\dot{\xi}_i)\frac{\partial \xi_i}{\partial q_s}$$

方程(3.18) 可称为变质量高阶非线性非完整系统广义坐标形式下的第二形式的广义 Volterra 方程。

如果 $m=1$，则(3.18)式成为

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial P_\sigma}-\frac{\partial \tilde{T}}{\partial \pi_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial \dot{q}_s}\left(\frac{d}{dt}\frac{\partial \dot{q}_s}{\partial P_\sigma}-\frac{\partial \dot{q}_s}{\partial \pi_\sigma}\right)=\tilde{E}_\sigma+\tilde{\Phi}'_\sigma \tag{3.19}$$

$$(\sigma=1,2,\cdots,\varepsilon)$$

其中

$$\tilde{E}_\sigma=\sum_{s=1}^{n}Q_s\frac{\partial \dot{q}_s}{\partial P_\sigma}\qquad \tilde{\Phi}'_\sigma=\sum_{s=1}^{n}\phi'_s\frac{\partial \dot{q}_s}{\partial P_\sigma}$$

方程(3.19)称为变质量一阶非完整系统广义坐标形式下的第二形式的广义 Volterra 方程.

如果 $m=1$，且 $m_i(t)=$ 常量，此时方程(3.19)可化为

$$\frac{d}{dt}\frac{\partial \widetilde{T}}{\partial P_\sigma}-\frac{\partial \widetilde{T}}{\partial \pi_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial \dot{q}_s}\left(\frac{d}{dt}\frac{\partial \dot{q}_s}{\partial P_\sigma}-\frac{\partial \dot{q}_s}{\partial \pi_\sigma}\right)=\widetilde{E}_\sigma \tag{3.20}$$

$$(\sigma=1,2,\cdots,g)$$

方程(3.20)与文献[2]的结果相同.

下面变换第一形式的广义 Volterra 方程.

将(3.9)和(3.12)代入第一形式的广义 Volterra 方程(2.12)，得到

$$\frac{d}{dt}\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{P_\sigma}}-\sum_{i=1}^{3N}m_i\dot{\xi}_i\left(\sum_{s=1}^{n}\frac{\partial \xi_i}{\partial q_s}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}+\sum_{s=1}^{n}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\frac{d}{dt}\frac{\partial \xi_i}{\partial q_s}\right)=\widetilde{E}_\sigma+\tilde{\psi}'_\sigma \tag{3.21}$$

注意到

$$\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q_s}}=\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}}\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m)}{q_s}}=\sum_{i=1}^{3N}m_i\dot{\xi}_i\frac{\partial \xi_i}{\partial q_s} \tag{3.22}$$

$$\begin{aligned}\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m-1)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}=&\sum_{s=1}^{n}\left[\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}}\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m-1)}{q_s}}+\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m-1)}{\xi_i}}\frac{\partial \overset{(m-1)}{\xi_i}}{\partial \overset{(m-1)}{q_s}}\right]\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&-m\sum_{s=1}^{n}\sum_{i=1}^{3N}m_i\dot{\xi}_i\frac{d}{dt}\left(\frac{\partial \xi_i}{\partial q_s}\right)\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\&+\sum_{i=1}^{3N}[(m-1)m_i\ddot{\xi}_i+(m-1)\dot{m}_i\dot{\xi}_i]\frac{\partial (\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}\end{aligned} \tag{3.23}$$

由于

$$\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{P_\sigma}}=\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q_s}}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}} \tag{3.24}$$

将上式两边对时间 t 求导，得

$$\begin{aligned}\frac{d}{dt}\frac{\partial \overset{(\widetilde{m-1})}{T}}{\partial \overset{(m-1)}{P_\sigma}}=&\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q_s}}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}+\sum_{s=1}^{n}\frac{d}{dt}\left[\sum_{i=1}^{3N}\frac{\partial \overset{(m-1)}{T_0}}{\partial \overset{(m)}{\xi_i}}\frac{\partial \overset{(m)}{\xi_i}}{\partial \overset{(m)}{q_s}}\right]\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}\\=&\sum_{s=1}^{n}\frac{\partial \overset{(m-1)}{T_1}}{\partial \overset{(m)}{q_s}}\frac{d}{dt}\frac{\partial \overset{(m)}{q_s}}{\partial \overset{(m-1)}{P_\sigma}}+\sum_{i=1}^{3N}m_i\ddot{\xi}_i\frac{\partial (\overset{(m)}{\xi_i})}{\partial \overset{(m-1)}{P_\sigma}}\end{aligned}$$

$$+\sum_{i=1}^{3N}\dot{m}_i\dot{\xi}_i\frac{\partial\overset{(m)}{(\xi_i)}}{\partial\overset{(m-1)}{P_\sigma}}+\sum_{s=1}^{n}\sum_{i=1}^{3N}m_i\dot{\xi}_i\frac{d}{dt}\left(\frac{\partial\overset{(m)}{\xi_i}}{\partial\overset{(m)}{q_s}}\right)\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}$$

于是

$$\begin{aligned}\sum_{i=1}^{3N}m_i\ddot{\xi}_i\frac{\partial\overset{(m)}{(\xi_i)}}{\partial\overset{(m-1)}{P_\sigma}}=&\frac{d}{dt}\frac{\partial\overset{(\widetilde{m-1})}{T}}{\partial\overset{(m-1)}{P_\sigma}}-\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T_1}}{\partial\overset{(m)}{q_s}}\frac{d}{dt}\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}\\&-\sum_{i=1}^{3N}\dot{m}_i\dot{\xi}_i\frac{\partial\overset{(m)}{(\xi_i)}}{\partial\overset{(m-1)}{P_\sigma}}-\sum_{s=1}^{n}\sum_{i=1}^{3N}m_i\dot{\xi}_i\\&\times\frac{d}{dt}\left(\frac{\partial\overset{(m)}{\xi_i}}{\partial\overset{(m)}{q_s}}\right)\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}\end{aligned}\tag{3.25}$$

将(3.25)代入(3.23)，整理、移项后，有

$$\begin{aligned}\sum_{i=1}^{3N}\sum_{s=1}^{n}m_i\dot{\xi}_i\frac{d}{dt}\left(\frac{\partial\overset{(m)}{\xi_i}}{\partial\overset{(m)}{q_s}}\right)\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}=&\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T_1}}{\partial\overset{(m-1)}{q_s}}\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}\\&-(m-1)\frac{d}{dt}\frac{\partial\overset{(\widetilde{m-1})}{T}}{\partial\overset{(m-1)}{P_\sigma}}+(m-1)\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T_1}}{\partial\overset{(m)}{q_s}}\frac{d}{dt}\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}\end{aligned}\tag{3.26}$$

最后，将(3.22)和(3.26)代入(3.21)，我们得到

$$m\frac{d}{dt}\frac{\partial\overset{(\widetilde{m-1})}{T}}{\partial\overset{(m-1)}{P_\sigma}}-m\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T_1}}{\partial\overset{(m)}{q_s}}\frac{d}{dt}\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}-\sum_{s=1}^{n}\frac{\partial\overset{(m-1)}{T_1}}{\partial\overset{(m-1)}{q_s}}\frac{\partial\overset{(m)}{q_s}}{\partial\overset{(m-1)}{P_\sigma}}=\tilde{E}_\sigma+\tilde{\Phi}'_\sigma$$

$$(\sigma=1,2,\cdots,\varepsilon)\tag{3.27}$$

方程 (3.27) 就是变质量高阶非线性非完整系统广义坐标形式下的第一形式的广义 Volterra 方程．

如果 $m=1$，则方程(3.27)可化为

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial P_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial\dot{q}_s}\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial P_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial q_s}\frac{\partial\dot{q}_s}{\partial P_\sigma}=\tilde{E}_\sigma+\tilde{\Phi}'_\sigma$$

$$(\sigma=1,2,\cdots,\varepsilon)\tag{3.28}$$

方程 (3.28) 就是变质量一阶非完整系统广义坐标形式下的第一形式的广义 Volterra 方程．

如果 $m=1$，且 $m_i(t)=$ 常量，此时方程(3.28)可化为

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial P_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial\dot{q}_s}\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial P_\sigma}-\sum_{s=1}^{n}\frac{\partial T_1}{\partial q_s}\frac{\partial\dot{q}_s}{\partial P_\sigma}=\tilde{E}_\sigma\ (\sigma=1,2,\cdots,\varepsilon)\tag{3.29}$$

方程(3.29)与文献[2]的结果相同．

本文曾得到北京工业学院梅凤翔教授的关心和支持，特此致谢．

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VOLTERRA'S EQUATION OF VARIABLE MASS IN HIGH-ORDER NONHOLONOMIC MECHANICAL SYSTEM.

Qiao Yongfen

(Northeast Agricultural College)

Abstract In this paper the Volterra's equtions of varlable mass in high-order nonlinear nonholonomic mechanical system are derived by the universal D'Alembert's principle of variable mass mechamical system.

Key words variable mass, higher order nonholonomic mechanical systems, generalized Volterra's equations.