第19卷 增刊
1987年7月

力 学 学 报
ACTA MECHANICA SINICA

Vol.19, SUP
July, 1987

变质量力学系统的拉格朗日方程

乔永芬
（东北农学院）

提要 本文从变质量力学系统的Kane方程，推出了变质量一阶非线性非完整系统的拉格朗日方程.

关键词 变质量力学系统；Kane方程；拉格朗日方程.

1. 变质量一阶非线性非完整力学系统广义坐标下的拉格朗日方程

1）用普通导数表示的形式

设有N个质量为$m_i(i=1,2,\cdots,N)$的质点所组成的变质量力学系统相对于惯性坐标系运动，其位形由n个广义坐标$q_r(r=1,2,\cdots,n)$确定.

设该系统受到g个理想非线性非完整约束

$$f_s(q_r,\ \dot{q}_r,\ t)=0 \qquad (s=P+1,P+2,\cdots,n) \tag{2.1}$$

的作用，其中$P=n-g$是系统的自由度.

现引入n个函数$W_s(s=1,2,\cdots,n)$如下：

$$W_s=\sum_{r=1}^{n} a_{sr}\dot{q}_r+b_s \qquad (s=1,2,\cdots,P) \tag{2.2}$$

$$W_s=f_s=0 \qquad (s=P+1,P+2,\cdots,n) \tag{2.3}$$

式中a_{sr}和b_s（$s=1,2,\cdots,P$；$r=1,2,\cdots,n$）是q_r和t的任意函数，但必须使得由(2.2)、(2.3)两式能够解出$\dot{q}_r(r=1,2,\cdots,n)$.为此目的，将(2.3)式写成如下形式：

$$W_s=\sum_{r=1}^{n}\frac{\partial f_s}{\partial \dot{q}_r}\dot{q}_r+h_s=0 \qquad (s=P+1,P+2,\cdots,n) \tag{2.4}$$

式中h_s不包含$\dot{q}_r$的线性项. 则，在(2.2)、(2.4)两式中的$a_{sr}(s=1,2,\cdots,P;\ r=1,2,\cdots,n)$和$\frac{\partial f_s}{\partial q_r}(s=P+1,P+2,\cdots,n;r=1,2,\cdots,n)$组成的行列式不等于零.于是可解出

$$\dot{q}_r=\sum_{s=1}^{P} k_{rs}W_s+l_r \qquad (r=1,2,\cdots,n) \tag{2.5}$$

由文献[1]可知，变质量系统中代表点的速度可写为

$$V_i=\sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}\cdot\dot{q}_r+\frac{\partial V_i}{\partial t} \tag{2.6}$$

现将(2.5)式代入(2.6)中，有

$$V_i=\sum_{s=1}^{P} U_{is}W_s+U_i \tag{2.7}$$

本文于1985年10月10日收到.

其中

$$U_{is}=\sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot q_r}k_{rs} \tag{2.8}$$

$$U_i=\sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot q_r}\cdot l_r+\frac{\partial V_i}{\partial t} \tag{2.9}$$

已知：

广义力

$$K_s=\sum_{i=1}^{N}F_i\cdot U_{is}=\sum_{r=1}^{n}\left[\sum_{i=1}^{N}F_i\cdot\frac{\partial V_i}{\partial \dot q_r}\right]k_{rs} \tag{2.10}$$

广义反推力

$$K_s^R=\sum_{i=1}^{N}R_i\cdot U_{is}=\sum_{r=1}^{n}\left[\sum_{i=1}^{N}R_i\cdot\frac{\partial V_i}{\partial \dot q_r}\right]k_{rs} \tag{2.11}$$

广义惯性力

$$K_s^*=\sum_{i=1}^{N}F_i^*\cdot U_{is}=-\sum_{r=1}^{n}\left[\sum_{i=1}^{N}m_i a_i\cdot\frac{\partial V_i}{\partial \dot q_r}\right]k_{rs} \tag{2.12}$$

现设系统中各质点的质量为 $m_i=m_i(q_r,\dot q_r,t)$

系统的动能

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i(V_i)^2$$

现暂不考虑非完整约束，不难得到

$$\sum_{i=1}^{N}m_i a_i\cdot\frac{\partial V_i}{\partial \dot q_r}=\frac{d}{dt}\frac{\partial T}{\partial \dot q_r}-\frac{\partial T}{\partial q_r}-\sum_{i=1}^{N}\dot m_i V_i\cdot\frac{\partial V_i}{\partial \dot q_r}-$$

$$-\frac{d}{dt}\left[\sum_{i=1}^{N}\frac{\partial m_i}{\partial \dot q_r}\frac{1}{2}(V_i)^2\right]+\sum_{i=1}^{N}\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2 \tag{2.13}$$

将上式代入(2.12)式中，得广义惯性力

$$K_s^*=-\sum_{r=1}^{n}\left\{\frac{d}{dt}\frac{\partial T}{\partial \dot q_r}-\frac{\partial T}{\partial q_r}-\sum_{i=1}^{N}\dot m_i V_i\cdot\frac{\partial V_i}{\partial \dot q_r}-\right.$$

$$\left.-\frac{d}{dt}\left[\sum_{i=1}^{N}\frac{\partial m_i}{\partial \dot q_r}\frac{1}{2}(V_i)^2\right]+\sum_{i=1}^{N}\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2\right\}k_{rs} \tag{2.14}$$

最后，我们将所求得的（2.10）、(2.11)和（2.14）式同时代入变质量力学系统的Kane方程[5]：

$$K_s+K_s^*+K_s^R=0 \qquad (s=1,2,\cdots,P) \tag{A}$$

即得

$$\sum_{r=1}^{n}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot q_r}-\frac{\partial T}{\partial q_r}\right]k_{rs}=K_s+\varphi_s \tag{2.15}$$

$$(s=1,2,\cdots,P)$$

其中

$$\varphi_s=\sum_{r=1}^{n}\left\{\sum_{i=1}^{N}\left[(R_i+\dot m_i V_i)\frac{\partial V_i}{\partial \dot q_r}+\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot q_r}\frac{1}{2}(V_i)^2\right)-\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2\right]\right\}k_{rs}$$

方程(2.15)即为变质量一阶非线性非完整力学系统广义坐标下的拉格朗日方程．

其次，若变质量力学系统受到g个理想线性非完整约束

$$\sum_{r=1}^{n} a_{sr}\dot{q}_r + b_s = 0 \qquad (s = P+1, P+2, \cdots, n) \tag{2.16}$$

的作用．此处$P = n - g$，其中a_{sr}和b_s一般是$q_r(r = 1, 2, \cdots, n)$和t的函数．因为这些方程都是彼此独立的，故可能出g个$\dot{q}_r$，即

$$\dot{q}_r = \sum_{i=1}^{P} C_{ri}\dot{q}_i + d_r \qquad (r = P+1, P+2, \cdots, n) \tag{2.17}$$

现引入n个函数$W_s(s = 1, 2, \cdots, n)$如下：

$$W_s = \sum_{r=1}^{n} a_{sr}\dot{q}_r + b_s \qquad (s = 1, 2, \cdots, n) \tag{2.18}$$

设（2.18）式中的最后g个方程（$s = P+1, P+2, \cdots, n$）是与方程（2.16）相同，而在其余方程中的a_{sr}和$b_s(s = 1, 2, \cdots, P; r = 1, 2, \cdots, n)$可允许任意给定．

规定　$\det(a_{sr}) \neq 0$　$W_s = 0$　$(s = P+1, P+2, \cdots, n)$

令a_{rs}^{-1}代表a_{rs}的逆矩阵的各元素，那么

$$\dot{q}_r = \sum_{s=1}^{n} a_{rs}^{-1}(W_s - b_s) \tag{2.19}$$

由此可得

$$\frac{\partial \dot{q}_r}{\partial W_s} = a_{rs}^{-1} \tag{2.20}$$

比较式(2.5)与(2.20)可知，当约束是一阶线性时，有

（1）$$k_{rs} = \frac{\partial \dot{q}_r}{\partial W_s} = a_{rs}^{-1}$$

（2）$$\begin{aligned}\varphi_s &= \sum_{r=1}^{n}\left\{\sum_{i=1}^{N}\left[(\boldsymbol{R}_i + \dot{m}_i\boldsymbol{V}_i)\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r} + \frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_r}\frac{1}{2}(\boldsymbol{V}_i)^2\right) - \frac{\partial m_i}{\partial q_r}\frac{1}{2}(\boldsymbol{V}_i)^2\right]\right\}k_{rs} \\ &= \sum_{r=1}^{n}\left\{\sum_{i=1}^{N}\left[(\boldsymbol{R}_i + \dot{m}_i\boldsymbol{V}_i)\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r} + \frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_r}\frac{1}{2}(\boldsymbol{V}_i)^2\right) - \frac{\partial m_i}{\partial q_r}\frac{1}{2}(\boldsymbol{V}_i)^2\right]\right\}a_{rs}^{-1} \\ &= \psi_s\end{aligned}$$

（3）$$K_s = \sum_{r=1}^{n}\left(\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}\right)k_{rs} = \sum_{r=1}^{n}\left(\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}\right)a_{rs}^{-1} = \widetilde{K}_s$$

于是方程（2.15）便成为

$$\sum_{r=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r}\right)a_{rs}^{-1} = \widetilde{K}_s + \psi_s \qquad (s = 1, 2, \cdots, P) \tag{2.21}$$

再讨论两种特殊情况：

情况1．　若取　$W_s = \dot{q}_s$　$(s = 1, 2, \cdots, P)$

则a_{rs}的$n\times n$阶矩阵A 取块形：

$$A=\left\|\begin{array}{c:c} I & 0 \\ \hdashline A_1 & A_2 \end{array}\right\|$$

其中I是$P\times P$阶单位矩阵，而A_1和A_2是$m\times P$和$m\times m$阶矩阵，其元素和方程（2.16）中$\dot{q}_r$的系数相同．

A的逆矩阵是

$$A^{-1}=\left\|\begin{array}{c:c} I & 0 \\ \hdashline -A_2^{-1}A_1 & A_2^{-1} \end{array}\right\|$$

其中A_2^{-1}是A_2的逆矩阵．定义C是$P\times m$矩阵$-A_2^{-1}A_1$，那么C的元素正是方程（2.17）中的系数C_{rs}．

现在方程（2.21）可化为如下形式

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}-\frac{\partial T}{\partial q_s}+\sum_{r=P+1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r}-\frac{\partial T}{\partial q_r}\right)C_{rs}=\widetilde{K}_s+\psi_s \tag{2.22}$$

$$(s=1,2,\cdots,P)$$

情况2．若动能　$T=T(\dot{q}_r,t)$

应用方程（2.18），有

$$\frac{\partial T}{\partial \dot{q}_r}=\sum_{k=1}^{n}\frac{\partial T}{\partial W_k}\frac{\partial W_k}{\partial \dot{q}_r}=\sum_{k=1}^{n}\frac{\partial T}{\partial W_k}a_{kr}\qquad (r=1,2,\cdots,n)$$

那么，方程（2.21）可化为

$$\frac{d}{dt}\frac{\partial T}{\partial W_s}+\sum_{r=1}^{n}\sum_{k=1}^{n}\frac{\partial T}{\partial W_k}\frac{da_{kr}}{dt}a_{rs}^{-1}=\widetilde{K}_s+\psi_s \tag{2.23}$$

$$(s=1,2,\cdots,P)$$

2）用凝固导数表示的形式

令$\frac{D}{Dt}$为质量不变时对时间的导数，$\frac{Д}{Д\dot{q}_r}$，$\frac{Д}{Дq_r}$分别为质量不变时对$\dot{q}_r$及对q_r的偏导数．由于

$$\begin{aligned}\boldsymbol{a}_i\cdot \boldsymbol{U}_{is}&=\boldsymbol{a}_i\cdot\sum_{r=1}^{n}\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}k_{rs}=\sum_{r=1}^{n}\left(\dot{\boldsymbol{V}}_i\cdot\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}\right)k_{rs}\\ &=\sum_{r=1}^{n}\left[\frac{d}{dt}\left(\boldsymbol{V}_i\cdot\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}\right)-\boldsymbol{V}_i\cdot\frac{d}{dt}\left(\frac{\partial \boldsymbol{V}_i}{\partial \dot{q}_r}\right)\right]k_{rs}\\ &=\sum_{r=1}^{n}\frac{1}{2}\left[\frac{d}{dt}\left(\frac{\partial \boldsymbol{V}_i^2}{\partial \dot{q}_r}\right)-\frac{\partial \boldsymbol{V}_i^2}{\partial q_r}\right]k_{rs}\end{aligned} \tag{2.24}$$

又不难得到

$$\frac{D}{Dt}\left(\frac{ДT}{Д\dot{q}_r}\right)-\frac{ДT}{Дq_r}=\sum_{i=1}^{N}\frac{m_i}{2}\left[\frac{d}{dt}\left(\frac{\partial \boldsymbol{V}_i^2}{\partial \dot{q}_r}\right)-\frac{\partial \boldsymbol{V}_i^2}{\partial q_r}\right] \tag{2.25}$$

于是，广义惯性力为

$$K_s^*=-\sum_{i=1}^{N}m_i\boldsymbol{a}_i\cdot \boldsymbol{U}_{is}=-\sum_{r=1}^{n}\left\{\sum_{i=1}^{N}\frac{m_i}{2}\left[\frac{d}{dt}\left(\frac{\partial \boldsymbol{V}_i^2}{\partial \dot{q}_r}\right)-\frac{\partial \boldsymbol{V}_i^2}{\partial q_r}\right]\right\}k_{rs}$$

$$= -\sum_{r=1}^{n}\left[\frac{D}{Dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}\right]k_{rs} \tag{2.26}$$

现将式(2.10)、(2.11)和(2.26)同时代入Kane方程(A)中，即得

$$\sum_{r=1}^{n}\left[\frac{D}{Dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}\right]k_{rs} = K_s + K_s^R \qquad (s=1,2,\cdots,P) \tag{2.27}$$

2. 变质量一阶非线性非完整力学系统准坐标下的拉格朗日方程

1) 用普通导数表示的形式

设变质量系统受g个理想非线性非完整约束

$$f_s(q_r,\ \dot{q}_r,\ t) = 0 \qquad (s = P+1, P+2, \cdots, n)$$

的作用．这里 $P = n - g$

假设取准速度如下：

$$\omega_s = \sum_{r=1}^{n} a_{sr}\dot{q}_r + b_s \qquad (s=1,2,\cdots,P) \tag{3.1}$$

$$\omega_s = \sum_{r=1}^{n} \frac{\partial f_s}{\partial \dot{q}_r}\dot{q}_r + h_s = 0 \qquad (s=P+1, P+2, \cdots, n) \tag{3.2}$$

现规定：a_{sr}和$\frac{\partial f_s}{\partial \dot{q}_r}$组成的行列式不等于零，于是可解出

$$\dot{q}_r = \sum_{s=1}^{P} k_{rs}\omega_s + l_r \qquad (r=1,2,\cdots,n) \tag{3.3}$$

将上式代入(2.6)中，有

$$V_i = \sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}\cdot\dot{q}_r + \frac{\partial V_i}{\partial t} = \sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}\left(\sum_{s=1}^{P}k_{rs}\omega_s + l_r\right) + \frac{\partial V_i}{\partial t}$$

$$= \sum_{s=1}^{P}\left(\sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}k_{rs}\right)\omega_s + \sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}\cdot l_r + \frac{\partial V_i}{\partial t} = \sum_{s=1}^{P}U_{is}\omega_s + U_i \tag{3.4}$$

其中

$$U_{is} = \sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}k_{rs} \qquad\qquad U_i = \sum_{r=1}^{n}\frac{\partial V_i}{\partial \dot{q}_r}\cdot l_r + \frac{\partial V_i}{\partial t}$$

已知：

广义力

$$K_s = \sum_{i=1}^{N} F_i\cdot U_{is} = \sum_{r=1}^{n}\left[\sum_{i=1}^{N} F_i\cdot\frac{\partial V_i}{\partial \dot{q}_r}\right]k_{rs} \tag{3.5}$$

广义反推力

$$K_s^R = \sum_{i=1}^{N} R_i\cdot U_{is} = \sum_{r=1}^{n}\left[\sum_{i=1}^{N} R_i\cdot\frac{\partial V_i}{\partial \dot{q}_r}\right]k_{rs} \tag{3.6}$$

广义惯性力

$$K_s^* = -\sum_{i=1}^{N} m_i a_i\cdot U_{is} = -\sum_{r=1}^{n}\left[\sum_{i=1}^{N} m_i a_i\cdot\frac{\partial V_i}{\partial \dot{q}_r}\right]k_{rs}$$

$$= -\sum_{r=1}^{n}\left\{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r}-\frac{\partial T}{\partial q_r}-\sum_{i=1}^{N}\dot{m}_i V_i\cdot\frac{\partial V_i}{\partial \dot{q}_r}-\right.$$

$$\left.-\frac{d}{dt}\left[\sum_{i=1}^{N}\frac{\partial m_i}{\partial \dot{q}_r}\frac{1}{2}(V_i)^2\right]+\sum_{i=1}^{N}\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2\right\}k_{rs} \tag{3.7}$$

现将式（3.5）、(3.6)和(3.7）同时代入Kane 方程（A）中，得到与方程（2.15）完全相同的方程，即

$$\sum_{r=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r}-\frac{\partial T}{\partial q_r}\right)k_{rs}=K_s+\varphi_s \qquad (s=1,2,\cdots,P)$$

现继续变换上式．令T^*为T中借助关系式(3.3)消去$\dot{q}_r$而表为ω_s的函数，有

$$\frac{\partial T^*}{\partial q_r}=\frac{\partial T}{\partial q_r}+\sum_{k=1}^{n}\frac{\partial T}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_r}$$

$$\sum_{r=1}^{n}\frac{\partial T^*}{\partial q_r}k_{rs}=\sum_{r=1}^{n}\frac{\partial T}{\partial q_r}k_{rs}+\sum_{r=1}^{n}\sum_{k=1}^{n}\frac{\partial T}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_r}k_{rs}$$

$$\frac{\partial T^*}{\partial \pi_s}=\sum_{r=1}^{n}\frac{\partial T}{\partial q_r}k_{rs}+\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\frac{\partial \dot{q}_r}{\partial \pi_s}$$

$$\frac{\partial T^*}{\partial \omega_s}=\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\frac{\partial \dot{q}_r}{\partial \omega_s}=\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}k_{rs}$$

$$\frac{d}{dt}\frac{\partial T^*}{\partial \omega_s}=\sum_{r=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)k_{rs}+\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\frac{dk_{rs}}{dt}$$

于是，有

$$\sum_{r=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r}-\frac{\partial T}{\partial q_r}\right)k_{rs}=\frac{d}{dt}\frac{\partial T^*}{\partial \omega_s}-$$

$$-\frac{\partial T^*}{\partial \pi_s}-\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\left(\frac{d}{dt}\frac{\partial \dot{q}_r}{\partial \omega_s}-\frac{\partial \dot{q}_r}{\partial \pi_s}\right) \tag{3.8}$$

将（3.8）式代入（3.7）中，得

$$K_s^*=-\left(\frac{d}{dt}\frac{\partial T^*}{\partial \omega_s}-\frac{\partial T^*}{\partial \pi_s}\right)+\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\left(\frac{d}{dt}\frac{\partial \dot{q}_r}{\partial \omega_s}-\frac{\partial \dot{q}_r}{\partial \pi_s}\right)+$$

$$-\left\{\sum_{i=1}^{N}\dot{m}_i V_i\cdot\frac{\partial V_i}{\partial \dot{q}_r}+\frac{d}{dt}\left[\sum_{i=1}^{N}\frac{\partial m_i}{\partial \dot{q}_r}\frac{1}{2}(V_i)^2\right]-\sum_{i=1}^{N}\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2\right\}k_{rs} \tag{3.9}$$

最后，把所求得的式（3.5）、(3.6）和（3.9）式同时代入Kane方程(A)中，有

$$\frac{d}{dt}\frac{\partial T^*}{\partial \omega_s}-\frac{\partial T^*}{\partial \pi_s}=K_s+\varphi_s+\sum_{r=1}^{n}\frac{\partial T}{\partial \dot{q}_r}\left(\frac{d}{dt}\frac{\partial \dot{q}_r}{\partial \omega_s}-\frac{\partial \dot{q}_r}{\partial \pi_s}\right) \qquad (s=1,2,\cdots,P) \tag{3.10}$$

其中

$$\varphi_s=\sum_{r=1}^{n}\left\{\sum_{i=1}^{N}\left[(R_i+\dot{m}_i V_i)\frac{\partial V_i}{\partial \dot{q}_r}+\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_r}\frac{1}{2}(V_i)^2\right)-\frac{\partial m_i}{\partial q_r}\frac{1}{2}(V_i)^2\right]\right\}k_{rs}$$

方程（3.10）就是我们得到的变质量一阶非线性非完整力学系统准坐标下的拉格朗日方程．

方程（3.10）比（2.15）更一般，因为当取准速度为广义速度时方程（3.10）便成为

(2.15).

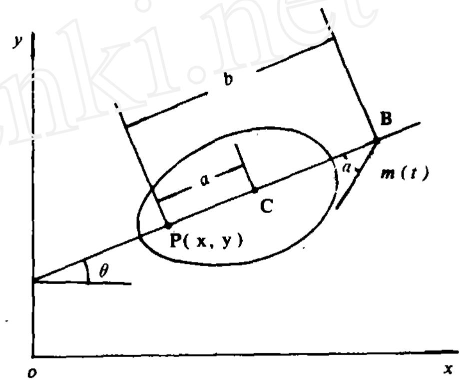
2）用凝固导数表示的形式

仿求方程（2.27）的步骤进行运算，即得

$$\frac{D}{Dt}\frac{\text{Д}T^*}{\text{Д}\omega_s}-\frac{\text{Д}T^*}{\text{Д}\pi_s}=K_s+K_s^R+\sum_{r=1}^{n}\frac{\text{Д}T}{\text{Д}\dot{q}_r}\left(\frac{d}{dt}\frac{\partial\dot{q}_r}{\partial\omega_s}-\frac{\partial\dot{q}_r}{\partial\pi_s}\right)\quad(s=1,2,\cdots,P)\qquad(3.11)$$

3. 算　例

在Чаплыгин-Caratheodory问题中物体对称轴PC上有一点B，其质量为$m=m(t)$．设物体质量为M，相对物体质心C的惯性矩为J_c．系统的位置由接触点的坐标(x,y)以及PC与ox间的夹角θ来确定．如图所示．设分离微粒的相对速度为v，试建立问题的运动微分方程．



解：　我们取物体及质点B为研究对象，主动力仅受重力作用，选x、y、θ为广义坐标．

约束方程　$\dot{y}=\dot{x}\,\text{tg}\,\theta$

而　$$\ddot{y}=\ddot{x}\,\text{tg}\,\theta+\frac{\dot{x}\dot{\theta}}{\cos^2\theta}\qquad(4.1)$$

本题属于变质量线性非完整系统．以x、θ为独立坐标．

令　$W_s=\dot{q}_s\qquad(s=1,2,)$

以q_1表示x、q_2表示θ、q_3表示y．于是可用公式（2.22）来解本题．但因$m=m(t)$，对本题来说，该公式可化为用凝固导数表示的如下形式：

$$\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}\dot{x}}-\frac{\text{Д}T}{\text{Д}x}+\left(\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}\dot{y}}-\frac{\text{Д}T}{\text{Д}y}\right)C_{31}=\widetilde{K}_1+K_1^R\qquad(4.2)$$

$$\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}\dot{\theta}}-\frac{\text{Д}T}{\text{Д}\theta}+\left(\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}\dot{y}}-\frac{\text{Д}T}{\text{Д}y}\right)C_{32}=\widetilde{K}_2+K_2^R\qquad(4.3)$$

由约束方程（4.1）可见，此处$C_{31}=\text{tg}\,\theta$　　$C_{32}=0$

系统的动能

$$T=\frac{1}{2}M[(\dot{x}-a\dot{\theta}\sin\theta)^2+(\dot{y}+a\dot{\theta}\cos\theta)^2]+\frac{1}{2}J_c\dot{\theta}^2$$

$$+\frac{1}{2}m(t)[(\dot{x}-b\dot{\theta}\sin\theta)^2+(\dot{y}+b\dot{\theta}\cos\theta)^2]$$

$$\frac{\text{Д}T}{\text{Д}x}=0\qquad\qquad\frac{\text{Д}T}{\text{Д}y}=0$$

$$\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}\dot{x}}=M[\ddot{x}-a\ddot{\theta}\sin\theta-a\dot{\theta}^2\cos\theta]+m(t)[\ddot{x}-b\ddot{\theta}\sin\theta-b\dot{\theta}^2\cos\theta]$$

$$\frac{D}{Dt}\frac{\text{Д}T}{\text{Д}y}=M[\ddot{x}\,\text{tg}\,\theta+\frac{\dot{x}\dot{\theta}}{\cos^2\theta}+a\ddot{\theta}\cos\theta-a\dot{\theta}^2\sin\theta]+$$

$$+m(t)\left[\ddot{x}\operatorname{tg}\theta+\frac{\dot{x}\dot{\theta}}{\cos^2\theta}+b\ddot{\theta}\cos\theta-b\dot{\theta}^2\sin\theta\right]$$

由于

$$V_c=(\dot{x}-a\dot{\theta}\sin\theta)i+(\dot{y}+a\dot{\theta}\cos\theta)j=(i+\operatorname{tg}\theta\cdot j)\dot{x}+(a\cos\theta\cdot j-a\sin\theta\cdot i)\dot{\theta}$$

所以偏速度

$$\frac{\partial V_c}{\partial\dot{x}}=i+\operatorname{tg}\theta\cdot j\qquad\qquad\frac{\partial V_c}{\partial\dot{\theta}}=a\cos\theta\cdot j-a\sin\theta\cdot i$$

又

$$V_B=(\dot{x}-b\dot{\theta}\sin\theta)i+(\dot{y}+b\dot{\theta}\cos\theta)j=(i+\operatorname{tg}\theta\cdot j)\dot{x}+(b\cos\theta\cdot j-b\sin\theta\cdot i)\dot{\theta}$$

$$\frac{\partial V_B}{\partial\dot{x}}=i+\operatorname{tg}\theta\cdot j\qquad\qquad\frac{\partial V_B}{\partial\dot{\theta}}=b\cos\theta\cdot j-b\sin\theta\cdot i$$

主动力

$$P=-Mg\boldsymbol{k}\qquad\qquad P(t)=-m(t)g\boldsymbol{k}$$

广义力

$$\widetilde{K}_1=(-Mgk)\cdot(i+\operatorname{tg}\theta\cdot j)+[-m(t)g\boldsymbol{k}]\cdot(i+\operatorname{tg}\theta\cdot j)=0$$

广义反推力

$$K_1^R=[\dot{m}u\cos(\alpha+\theta)i+\dot{m}u\sin(\alpha+\theta)j]\cdot[i+\operatorname{tg}\theta\cdot j]=\dot{m}u\frac{\cos\alpha}{\cos\theta}$$

这样方程（4.2）给出

$$\frac{1}{\cos^2\theta}[M+m(t)](\ddot{x}+\dot{x}\dot{\theta}\operatorname{tg}\theta)-\frac{\dot{\theta}^2}{\cos\theta}[Ma+m(t)b]=\dot{m}u\frac{\cos\alpha}{\cos\theta}\qquad(4.4)$$

$$\frac{D}{Dt}\frac{ДT}{Д\dot{\theta}}=M[(-\ddot{x}a\sin\theta+a^2\ddot{\theta}\sin^2\theta+a^2\dot{\theta}^2\cos\theta\sin\theta+\ddot{x}a\sin\theta+\frac{a\dot{x}\dot{\theta}}{\cos\theta}+a^2\ddot{\theta}\cos^2\theta-$$

$$-a^2\dot{\theta}^2\cos\theta\sin\theta-\dot{x}\dot{\theta}a^2\sin\theta\operatorname{tg}\theta-a^2\dot{\theta}^2\sin\theta\cos\theta+a^2\dot{\theta}^2\cos\theta\sin\theta-\ddot{x}a\dot{\theta}\cos\theta]+$$

$$+J_c\ddot{\theta}+m(t)[-\ddot{x}b\cos\theta+b^2\ddot{\theta}\sin^2\theta-b^2\dot{\theta}^2\cos\theta\sin\theta-b\dot{x}\dot{\theta}\cos\theta+b\ddot{x}\sin\theta+$$

$$+b^2\dot{\theta}^2\sin\theta\cos\theta+b^2\ddot{\theta}\cos^2\theta-b^2\dot{\theta}^2\sin\theta\cos\theta-b\dot{x}\dot{\theta}\sin\theta\operatorname{tg}\theta-b^2\dot{\theta}^2\sin\theta\cos\theta+\frac{b\dot{x}\dot{\theta}}{\cos\theta}]$$

$$\frac{ДT}{Д\theta}=M[(\dot{x}-a\dot{\theta}\sin\theta)(-a\dot{\theta}\cos\theta)+(\dot{y}+a\dot{\theta}\cos\theta)(-a\dot{\theta}\sin\theta)]+m(t)[(\dot{x}-b\dot{\theta}\sin\theta)\times$$

$$\times(-b\dot{\theta}\cos\theta)+(\dot{y}+b\dot{\theta}\cos\theta)(-b\dot{\theta}\sin\theta)]$$

广义力

$$\widetilde{K}_2=(-Mg\boldsymbol{k})\cdot(a\cos\theta\cdot j-a\sin\theta\cdot i)+[-m(t)g\boldsymbol{k}]\cdot(b\cos\theta\cdot j-b\sin\theta\cdot i)=0$$

广义反推力

$$R_\theta=\dot{m}ub\sin\alpha\cdot\boldsymbol{k}\qquad\qquad\omega=\dot{\theta}\boldsymbol{k}\qquad\qquad\frac{\partial\boldsymbol{\omega}}{\partial\dot{\theta}}=\boldsymbol{k}$$

$$K_2^R = (\dot{m} u b \sin\alpha \cdot k) \cdot k = \dot{m} u b \sin\alpha$$

于是方程（4.3）给出

$$[J_c + Ma^2 + m(t)b^2]\ddot{\theta} + \frac{\dot{x}\dot{\theta}}{\cos\theta}[Ma + m(t)b] = \dot{m} u b \sin\alpha \tag{4.5}$$

方程（4.4）、（4.5）即为所求方程.

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LAGRANGE'S EQUATIONS FOR THE MECHANICAL SYSTEMS HAVING VIRIABLE MASS

Qiao Yongfen

(Northeast agricultural college)

ABSTRACT In this paper the lagrange's equations for the first order nonlinear non-holonomic mechanical systems having variable mass was deduced from the T. R. Kane's equation for the mechanical systems having variable mass.

KEY WORDS mechanical systems, variable mass, Kane's equation, lagrange's equation.