弹性和粘弹性力学的一组变分原理*

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提要 本文建立了一个反映线弹性力学的所有场方程（包括余能型应力应变关系）和边界条件的新变分原理，并由该原理的推广而引出弹性、粘弹性力学线性和非线性理论的一组新变分原理。

一、线弹性静力学的变分原理

有序函数组，〔U_i，ε_{ij}，σ_{ij}〕为线弹性静力学的真实状态的充要条件为：泛函 Π^{LS} 在〔U_i，ε_{ij}，σ_{ij}〕处取驻值，这里

$$\Pi^{LS}=\int_R\left[\widetilde{W}\left(\varepsilon_{ij}-\frac{\partial\widetilde{W}}{\partial\sigma_{ij}}\right)\left(\varepsilon_{ij}-\frac{\partial\widetilde{W}}{\partial\sigma_{ij}}\right)+\sigma_{ij}(U_{i,j}+U_{j,i})/2-\widetilde{W}-F_iU_i\right]dV$$

$$-\int_{\partial R_u}T_i(U_i-U_i)dA-\int_{\partial R_t}\overline{T}_iU_idA$$

其中，$\widetilde{W}$ 是余应变能函数。

二、线弹性动力学的变分原理

有序函数组〔U_i，ε_{ij}，σ_{ij}〕为线弹性动力学的真实状态的充要条件为：泛函 Π^{LD} 在〔U_i，ε_{ij}，σ_{ij}〕处取驻值，这里

$$\Pi^{LD}=\int_R\left[-g*\frac{\partial\widetilde{W}}{\partial\sigma_{ij}}*\sigma_{ij}*\left(\varepsilon_{rs}-\frac{\partial\widetilde{W}}{\partial\sigma_{rs}}\right)*\left(\varepsilon_{rs}-\frac{\partial\widetilde{W}}{\partial\sigma_{rs}}\right)+g*\sigma_{ij}*(U_{i,j}\right.$$

$$\left.+U_{j,i})/2-g*\frac{\partial\widetilde{W}}{\partial\sigma_{ij}}*\sigma_{ij}/2-\overline{F}_i*U_i+\rho U_i*U_i/2\right]dV$$

$$-\int_{\partial R_t}g*\overline{T}_i*U_idA-\int_{\partial R_u}g*T_i*(U_i-\overline{U}_i)dA$$

其中

$$g(t)=\begin{cases}0 & t\in(-\infty,\ 0)\\ t & t\in[0,\ \infty)\end{cases}$$

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* 作者在第 16 届 IUTAM 大会上的报告全文将发表于 Journal de Mécanique。

$$\bar{F}_i=g*F_i+\rho(t\dot{U}_i|_{t=0}+U_i|_{t=0})$$

而符号 * 表示函数关于时间参数 t 的卷积，如

$$g*f=\int_0^t g(s)f(t-s)ds$$

三、线粘弹性动力学的变分原理

在集合 E_V 上，有序函数组〔U_i，ε_{ij}，σ_{ij}〕为线粘弹性动力学的真实状态的充要条件为：泛函 Π^V 在〔U_i，ε_{ij}，σ_{ij}〕处取驻值，这里

$$E_V=\{[U_i,\ \varepsilon_{ij},\ \sigma_{ij}]/[U_i,\ \varepsilon_{ij},\ \sigma_{ij}]=[U_i^l,\ \varepsilon_{ij}^l,\ \sigma_{ij}^l]\ t\in(-\infty,0)\}$$

$$\Pi^V=\int_R\Big[-\frac{1}{2}J_{mnpq}*\sigma_{mn}*\sigma_{pq}*(g*\varepsilon_{ij}-l*J_{ijkl}*\sigma_{kl}-g*\varepsilon_{ij}^{+})*(g*\varepsilon_{ij}$$
$$-l*J_{ijkl}*\sigma_{kl}-g*\varepsilon_{ij}^{+})+g*\sigma_{ij}*\varepsilon_{ij}^{+}+g*\sigma_{ij}*(U_{i,j}+U_{j,i})/2$$
$$-l*J_{ijkl}*\sigma_{ij}*\sigma_{kl}/2-\bar{F}_i*U_i+\rho U_i*U_i/2\Big]dV-\int_{\partial R_t}g*\bar{T}_i*U_i dA$$
$$-\int_{\partial R_u}g*T_i*(U_i-\bar{U}_i)dA$$

其中

$$l(t)=\begin{cases}0 & t\in(-\infty,\ 0)\\ 1 & t\in[0,\ \infty)\end{cases}$$

$$\dot{\varepsilon}_{ij}^{+}=\int_0^{\infty}\sigma_{rs}^l(x,s)\dot{J}_{ijrs}(x,t-s)ds$$

而 U_i^l，ε_{ij}^l，σ_{ij}^l 是已知函数。

四、非线性弹性静力学的变分原理

1．全量型变分原理

有序函数组〔U_a，x_{aA}，T_{Aa}〕为非线性弹性静力学的真实状态的充要条件为：泛函 Π^{NS} 在〔U_a，x_{aA}，T_{Aa}〕处取驻值，这里

$$\Pi^{NS}=\int_R\Big[-\widetilde{W}+T_{Aa}x_{a,A}+\widetilde{W}\Big(x_{aA}-\frac{\partial\widetilde{W}}{\partial T_{Aa}}\Big)\Big(x_{aA}-\frac{\partial\widetilde{W}}{\partial T_{Aa}}\Big)-\rho_0F_aU_a\Big]dV$$
$$-\int_{\partial R_t}\bar{T}_aU_a dA-\int_{\partial R_u}T_a(U_a-\bar{U}_a)dA$$

2．增量型变分原理

有序函数组〔U_a+u_a，$x_{aA}+u_{aA}$，$T_{Aa}+t_{Aa}$〕为非线性弹性静力学的真实状态的充要条件为：泛函 Π_I^{NS} 在〔U_a+u_a，$x_{aA}+u_{aA}$，$T_{Aa}+t_{Aa}$〕处取驻值，这里

$$\Pi_I^{NS}=\int_R\Big[\Big(x_{aA}+u_{aA}-\frac{\partial\widetilde{W}}{\partial T_{Aa}}-\frac{\partial^2\widetilde{W}}{\partial T_{Aa}\partial T_{Bb}}t_{Bb}\Big)\Big(x_{aA}+u_{aA}-\frac{\partial\widetilde{W}}{\partial T_{Aa}}$$
$$-\frac{\partial^2\widetilde{W}}{\partial T_{Aa}\partial T_{Bb}}t_{Bb}\Big)\widetilde{W}+(T_{Aa}+t_{Aa})(x_{a,A}+u_{a,A})-\frac{\partial\widetilde{W}}{\partial T_{Aa}}t_{Aa}-\frac{\partial^2\widetilde{W}}{\partial T_{Aa}\partial T_{Bb}}t_{Aa}t_{Bb}/2$$

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Some Results in Mechanics of Catastrophic Multiphase Phenomena

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Abstract

Analysis for catastrophic multiphase phenomena is important for engineering purpose. Such phenomena are known as that of fracture (solid-gas), cavitation (fluid-gas) or both (solid-fluid-gas). Up to now, research on these problems are done separatively in methodology. In this paper we discuss such problems for finite fracture dynamics of nonlinear media and cavitation dynamics of elastic fluids with the common methodology of path-independent integrals[1]. Some path-independent integrals are worked out and related physical meaning for them are given as the extension force.

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$$-\rho_0(F_a+f_a)(U_a+u_a)\Big]dV-\int_{\partial R_t}(\overline{T}_a+\overline{t}_a)(U_a+u_a)dA$$

$$-\int_{\partial R_u}(T_a+t_a)(U_a+u_a-\overline{U}_a-\overline{u}_a)dA$$

其中，u_a，u_{aA}，t_{Aa}，f_a 和 t_a 分别为 U_a，x_{aA}，T_{Aa}，F_a 和 T_a的增量。

泛函中的四阶项会增加数值计算的困难，如何据上述原理建立简便的计算格式，乃是有待研究的课题。

A Group of Variational Principles in Elasticity and Viscoelasticity

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Abstract

In this paper, a new variational principle which reflects all the field equations (involving the stress-strain relations in complementary energy form) and the boundary conditions in linear elastostatics is established first. Then, this variational principle is generalized and a group of new variational principles in linear and nonlinear theories of elasticity and viscoelasticity is obtained.