非线性随机振动分析的谱分解法

——与当前通用方法的比较

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提要 当前国际上用以分析非线性随机振动的有F．P．概率方程法，等价线性化法以及摄动法。这些方法都有较大的局限性，而且计算复杂，缺少一定的数值计算方法。本文对其作了评述以后，将随机过程进行谱分解，按两种不同的统计规率提出两种新的非线性随机振动的分析方法。结合有限元法[1]，用数值方法求解，本文方法可以广泛地应用在各种非线性随机过程领域。

一、当前非线性随机振动分析法的评述

1．F．P．概率方程方法[2]、[3]

这种方法用在 Markov 过程，一般用在伊藤随机微分方程，要求激励是平稳 Gauss 白噪声过程，对一些实测的非常数功率谱不能适用，求解析解困难，也缺少有效的数值方法。但是值得提出的是本方法如可解时则是精确解，可用来和其它方法比较。今有随机过程：

$$\ddot{Z}(t)+2\beta\dot{Z}(t)+\omega_0^2[Z(t)+aZ^3(t)]=X(t) \qquad (1)$$

$X(t)$ 是期望值为零的平稳 Gauss 白噪声过程。今设 Z，$\dot{Z}$ 的稳定概率密度函数是 $f_s(z,\dot{z})$，可以求出相应的 FP 方程是：

$$D\frac{\partial^2 f_s}{\partial \dot{z}^2}-\frac{\partial}{\partial z}(\dot{z}f_s)+\frac{\partial}{\partial \dot{z}}\{[k(z)+2\beta\dot{z}]f_s\}=0$$

这里，$k(z)=\omega_0^2[z(t)+az^3(t)]$，用分离变量法求解得：

$$\left.\begin{aligned}
&f(z,\dot{z})=f_1(\dot{z})f_2(z)\\
&f_1(\dot{z})=C_1\exp\left(-\frac{\beta}{D}\dot{z}^2\right)\\
&f_2(z)=C_2\exp\left\{-\frac{2\beta}{D}\left(\frac{\omega_0^2z^2}{2}+\frac{a\omega_0^2z^4}{4}\right)\right\}
\end{aligned}\right\} \qquad (2)$$

用归一化条件决定 C_1 和 C_2。再用公式：

$\sigma_Z^2=\int_{-\infty}^{\infty}z^2f_2(z)dz$ 求 Z 的均方值 σ_Z^2。如果 a 是小量，将 σ_Z^2 展开成 a 的幂级数可得：

$$\sigma_Z^2=\sigma_{Z0}^2(1-3a\sigma_{Z0}^2+24a^2\sigma_{Z0}^4+\cdots) \qquad (3)$$

再考虑 $a\to\infty$ 时的渐近性质，由于

$$\sigma_Z^2=\int_{-\infty}^{\infty}z^2 f_2(z)\,dz\Big/\int_{-\infty}^{\infty}f_2(z)dz \tag{4}$$

引入新变量 η，使$\frac{\beta a\omega_0^2 z^4}{S_0}=\eta^4$，$S_0=2D$ 是白噪声功率谱，注意到：$\sigma_{Z0}^2=\frac{S_0}{2\beta\omega_0^2}$，则有

$$\frac{a}{2\sigma_{Z0}^2}z^4=\eta^4,\ z^4=\frac{2\sigma_{Z0}^2}{a}\eta^4,\quad z=\left(\frac{2}{a}\right)^{1/4}\sqrt{\sigma_{Z0}}\,\eta \tag{5}$$

$$\sigma_Z^2=\sqrt{\frac{2}{a}}\sigma_{Z0}\int_{-\infty}^{\infty}\eta^2 e^{-\eta^4}d\eta\Big/\int_{-\infty}^{\infty}e^{-\eta^4}d\eta=\sqrt{\frac{2}{a}}\sigma_{Z0}\frac{\Gamma(3/4)}{\Gamma(1/4)}=\frac{0.477\sigma_{Z0}}{\sqrt{a}} \tag{6}$$

可见当 $a\to\infty$ 时，σ_Z^2 按$\sqrt{a}$的比例趋向于零。

2．等价线性化法

激励不限于白噪声，$S_X(\omega)$ 可具有任意形式，是计算非线性随机问题的一种实用方法，如果补充一些数值计算程序可推广其应用范围。

（1）式的等价线性化方程是：

$$\ddot{Z}(t)+2\beta_e\dot{Z}(t)+\omega_e^2 Z(t)=X(t) \tag{7}$$

根据等价条件可得：

$$\beta_e=\beta,\ \omega_e^2=\omega_0^2(1+3a\sigma_z^2) \tag{8}$$

$$\left.\begin{aligned}&\sigma_Z^2=\int_{-\infty}^{\infty}S_Z(\omega)d\omega=\int_{-\infty}^{\infty}|G(\omega)|^2 S_X(\omega)d\omega\\&G(\omega)=(\omega_e^2-\omega^2+2i\beta\omega)^{-1}\end{aligned}\right\} \tag{9}$$

当激励是白噪声过程时从（9）式可求出：

$$\sigma_Z^2=\sigma_{Z0}^2\frac{1}{1+3a\sigma_Z^2}\ \text{或}\ 3a\sigma_Z^4+\sigma_Z^2-\sigma_{Z0}^2=0 \tag{10}$$

$$\sigma_Z^2=\frac{-1+\sqrt{1+12a\sigma_{Z0}^2}}{6a\sigma_{Z0}^2}\sigma_{Z0}^2 \tag{11}$$

当 a 较小时将（11）式按 a 的幂展开为：

$$\sigma_Z^2=\sigma_{Z0}^2(1-3a\sigma_{Z0}^2+18a^2\sigma_{Z0}^4-\cdots) \tag{12}$$

当 $a\to\infty$ 时 σ_Z^2 的渐近形式是

$$\sigma_Z^2=\frac{1}{\sqrt{3}}\frac{\sigma_{Z0}}{\sqrt{a}}=\frac{0.577\sigma_{Z0}}{\sqrt{a}} \tag{13}$$

与 F．P．方程法比较，在（12）式中 a^2的系数和（13）式的系数都发现离差。离差的原因是 F．P．方程法和等价线性化法对于激励和响应随机过程的统计性质作了不同的假定。F．P．方程法假定激励是 Gauss 过程，则响应 $Z(t)$ 由于非线性关系必不是 Gauss 过程；而从（7）式可见等价线性化法则假定激励和响应都是 Gauss 过程，这样出现离差都是理所当然的。理论上说，在激励和响应中只能选取一种作为 Gauss 过程，选取方法是首先考虑根据实际观测资料哪一种比较符合 Gauss 过程，其次考虑如何选取方便于计算。本文所提出的方法将分别考虑两种不同的统计假定。

3．摄动法

摄动法用在弱非线性问题，按照摄动级求解。一般假定激励是 Gauss 平稳过程，但不一定是白噪声过程，$S_X(\omega)$ 可以是 ω 的任意函数。对于高阶摄动项由于统计性质渐趋复杂，求解非常困难。仍以（1）式为例将 $Z(t)$ 展开为

$$Z(t)=Z_0(t)+aZ_1(t)+a^2Z_2(t)+\cdots \tag{14}$$

$$Z^2(t)=Z_0^2(t)+2aZ_0(t)Z_1(t)+a^2[Z_1^2(t)+2Z_0(t)Z_2(t)]+\cdots$$

$$\begin{aligned}\sigma_Z^2=E[Z^2(t)]&=E\{Z_0^2(t)+2aZ_0(t)Z_1(t)+a^2[Z_1^2(t)+2Z_0(t)Z_2(t)]+\cdots\}\\&=E[Z_0^2(t)]+2aE[Z_0(t)Z_1(t)]+a^2E[Z_1^2(t)+2Z_0(t)Z_2(t)]+\cdots\end{aligned} \tag{15}$$

将（14）式代入（1）式按摄动法可得如下一系列方程：

$$\left.\begin{aligned}&\ddot{Z}_0(t)+2\beta\dot{Z}_0(t)+\omega_0^2Z_0(t)=X(t)\\&\ddot{Z}_1(t)+2\beta\dot{Z}_1(t)+\omega_0^2Z_1(t)=-\omega_0^2Z_0^3(t)\ \text{等}\end{aligned}\right\} \tag{16}$$

求解上式考虑到 a 的一次项可得：

$$\begin{aligned}\sigma_Z^2=E[Z^2(t)]&\approx E[Z_0^2(t)]+2aE[Z_0(t)Z_1(t)]\\&=\sigma_{Z_0}^2\left[1-6a\omega_0^2\int_0^{\infty}h(s)\Gamma_{Z_0Z_0}(s)ds\right]\end{aligned} \tag{17}$$

式中 $h(t)$ 是脉冲反应函数，$\Gamma_{Z_0Z_0}(s)$ 是 Z_0 的相关函数，当激励 $X(t)$ 是白噪声过程时可以求得

$$\sigma_Z^2\approx\sigma_{Z_0}^2(1-3a\sigma_{Z_0}^2) \tag{18}$$

由于计算的复杂性，作者尚未看到在摄动法中二级近似的计算。由于摄动法和 F. P. 方程法的假定相同，可以断定摄动法的各级解应与 F. P. 方程法所得解相当的展开项相等。本文所提第一种方法实质上和摄动法相同，将在以后证明，当 σ_Z^2 展开到 a 的平方项时有：

$$\sigma_Z^2\approx\sigma_{Z_0}^2(1-3a\sigma_{Z_0}^2+24a^2\sigma_{Z_0}^4) \tag{19}$$

和 F. P. 方程法所得结果相同。

二、谱 分 解 法

1. 基本原理

在线性随机方程中谱分解法已有了广泛应用[4],[5]，作者将它推广到非线性情况。今设激励 X 和响应 Z 都是平稳二阶矩过程。为便于求解假设 X 和 Z 之一是 Gauss 过程。今有：

$$X(t)=\int_{-\infty}^{\infty}e^{it\omega}d\xi(\omega) \tag{20}$$

其中 $\{\xi(\omega),\ -\infty<\omega<\infty\}$ 是正交增量过程。为方便计算，激励 X 和响应 Z 可以形式地写为

$$X(t)=\int_{-\infty}^{\infty}\tilde{X}(\omega)e^{it\omega}d\omega \tag{21a}$$

$$Z(t)=\int_{-\infty}^{\infty}\tilde{Z}(\omega)e^{it\omega}d\omega \tag{21b}$$

$\tilde{X}(\omega)$，$\tilde{Z}(\omega)$ 分别称为过程 X，Z 的谱分解量，正交增量性质可以表示为如下公式：

$$\left.\begin{aligned}E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]=\delta(\omega_1+\omega_2)S_X(\omega_1)\\E[\tilde{Z}(\omega_1)\tilde{Z}(\omega_2)]=\delta(\omega_1+\omega_2)S_Z(\omega_1)\end{aligned}\right\} \tag{22}$$

式中 E 示求期望值，δ 示 Dirac 函数，$S_X(\omega_1)$ 和 $S_Z(\omega_1)$ 各为 X,Z 的功率谱函数。一般假定：

$$E[\tilde{X}(\omega)]=0,\ E[\tilde{Z}(\omega)]=0 \tag{23}$$

对高次乘积期望值有两种统计规律假定，第一种假设激励是 Gauss 过程，则有[2,3]

$$E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)]=E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_2)]E[\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)]$$
$$+E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_3)]E[\widetilde{X}(\omega_2)\widetilde{X}(\omega_4)]+E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_4)]E[\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)] \tag{24a}$$

$$\begin{aligned}E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)\widetilde{X}(\omega_5)\widetilde{X}(\omega_6)]\\ =E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_2)]E[\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)\widetilde{X}(\omega_5)\widetilde{X}(\omega_6)]\\ +E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_3)]E[\widetilde{X}(\omega_4)\widetilde{X}(\omega_5)\widetilde{X}(\omega_6)\widetilde{X}(\omega_2)]\\ +E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_4)]E[\widetilde{X}(\omega_5)\widetilde{X}(\omega_6)\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)]\\ +E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_5)]E[\widetilde{X}(\omega_6)\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)]\\ +E[\widetilde{X}(\omega_1)\widetilde{X}(\omega_6)]E[\widetilde{X}(\omega_2)\widetilde{X}(\omega_3)\widetilde{X}(\omega_4)\widetilde{X}(\omega_5)]\end{aligned} \tag{24b}$$

$\widetilde{X}(\omega_i)$ 的奇数个乘积的期望值为零。第二种假设响应是 Gauss 过程，则有：

$$E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)]=E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_2)]E[\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)]$$
$$+E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_3)]E[\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_4)]+E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_4)]E[\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)] \tag{25a}$$

$$\begin{aligned}E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)\widetilde{Z}(\omega_5)\widetilde{Z}(\omega_6)]\\ =E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_2)]E[\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)\widetilde{Z}(\omega_5)\widetilde{Z}(\omega_6)]\\ +E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_3)]E[\widetilde{Z}(\omega_4)\widetilde{Z}(\omega_5)\widetilde{Z}(\omega_6)\widetilde{Z}(\omega_2)]\\ +E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_4)]E[\widetilde{Z}(\omega_5)\widetilde{Z}(\omega_6)\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)]\\ +E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_5)]E[\widetilde{Z}(\omega_6)\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)]\\ +E[\widetilde{Z}(\omega_1)\widetilde{Z}(\omega_6)]E[\widetilde{Z}(\omega_2)\widetilde{Z}(\omega_3)\widetilde{Z}(\omega_4)\widetilde{Z}(\omega_5)]\end{aligned} \tag{25b}$$

$\widetilde{Z}(\omega_i)$ 的奇数个乘积的期望值为零。本文所提两种解法，一种以（24a，b）式为基础，按 a 的次数求各级近似值，应和摄动法求得相同的解。第二种以（25a，b）式为基础，适用于任何非线性程度，通过适当的数值方法可求出封闭解。以下求两过程 $U(t)$，$V(t)$ 乘积的谱量。以下公式中积限 $\pm\infty$ 略去不写，星号 $*$ 表示搭积，今有

$$U(t)V(t)=\iint\widetilde{U}(\omega_1)\widetilde{V}(\omega-\omega_1)e^{i\omega t}d\omega_1 d\omega \tag{26a}$$

$$\widetilde{U(t)V(t)}=\int\widetilde{U}(\omega_1)\widetilde{V}(\omega-\omega_1)d\omega_1=\widetilde{U}(\omega)*\widetilde{V}(\omega) \tag{26b}$$

$$\sigma_Z^2=\int S_Z(\omega_1)d\omega_1 \tag{27a}$$

$$S_Z(\omega)*S_Z(\omega)=\int S_Z(\omega_1)S_Z(\omega-\omega_1)d\omega_1 \tag{27b}$$

$$S_Z(\omega)*S_Z(\omega)*S_Z(\omega)=\iint S_Z(\omega_2)S_Z(\omega_1-\omega_2)S_Z(\omega-\omega_1)d\omega_1 d\omega_2 \tag{27c}$$

2．第一种谱分解法

（1）单自由度情况

用变分原理研究结构动力学问题常可遇到如下类型公式：

$$\begin{aligned}&m\ddot{Z}+c_1\dot{Z}+k_1Z+k_2Z^2+k_3Z^3+c_2\dot{Z}Z+c_3\dot{Z}Z^2\\ &=m\ddot{Z}+c_1\dot{Z}+k_1Z+k_2Z^2+k_3Z^3+c_2\frac{d}{dt}\left(\frac{Z^2}{2}\right)+c_3\frac{d}{dt}\left(\frac{Z^3}{3}\right)=X\end{aligned} \tag{28}$$

假定非线性程序较小，将 Z 展开得一系列方程：

$$Z(t)=Z_0(t)+Z_1(t)+Z_2(t)+\cdots \tag{29}$$

$$m\ddot{Z}_0+c_1\dot{Z}_0+k_1Z_0=X \tag{30a}$$

$$m\ddot{Z}_1+c_1\dot{Z}_1+k_1Z_1=-k_2Z_0^2-k_3Z_0^3-\frac{c_2}{2}\frac{d}{dt}(Z_0^2)-\frac{c_3}{3}\frac{d}{dt}(Z_0^3) \tag{30b}$$

$$m\ddot{Z}_2+c_1\dot{Z}_2+k_1Z_2=-2k_2Z_0Z_1-3k_3Z_0^2Z_1-\frac{c_2}{2}\frac{d}{dt}(2Z_0Z_1)-\frac{c_3}{3}\frac{d}{dt}(3Z_0^2Z_1)\text{等} \tag{30c}$$

将以上三式写成谱分解形式，引入 $G(\omega)$ 则有：

$$G(\omega)=\frac{1}{-m\omega^2+c_1i\omega+k_1}=\frac{(k_1-m\omega^2)-c_1i\omega}{(k_1-m\omega^2)^2+c_1^2\omega^2} \tag{31}$$

$$\tilde{Z}_0(\omega)=G(\omega)\tilde{X}(\omega) \tag{32a}$$

$$\begin{aligned}\tilde{Z}_1(\omega)=&-G(\omega)[k_2\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]\\&-G(\omega)i\omega\left[\frac{c_2}{2}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+\frac{c_3}{3}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)\right]\end{aligned} \tag{32b}$$

$$\begin{aligned}\tilde{Z}_2(\omega)=&-G(\omega)[2k_2\tilde{Z}_0(\omega)*\tilde{Z}_1(\omega)+3k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_1(\omega)]\\&-G(\omega)i\omega[c_2\tilde{Z}_0(\omega)*\tilde{Z}_1(\omega)+c_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_1(\omega)]\\=&G(\omega)\{2k_2\tilde{Z}_0(\omega)*G(\omega)[k_2\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]\}\\&+iG(\omega)\left\{2k_2\tilde{Z}_0(\omega)*\omega G(\omega)\left[\frac{c_2}{2}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+\frac{c_3}{3}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)\right]\right\}\\&+G(\omega)\{3k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*G(\omega)[k_2\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]\}\\&+iG(\omega)\left\{3k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\omega G(\omega)\left[\frac{c_2}{2}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+\frac{c_3}{3}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)\right]\right\}\\&+i\omega G(\omega)\{c_2\tilde{Z}_0(\omega)*G(\omega)[k_2\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]\}\\&-\omega G(\omega)\left\{c_2\tilde{Z}_0(\omega)*\omega G(\omega)\left[\frac{c_2}{2}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+\frac{c_3}{3}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)\right]\right\}\\&+i\omega G(\omega)\{c_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*G(\omega)[k_2\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+k_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]\}\\&-\omega G(\omega)\left\{c_3\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\omega G(\omega)\left[\frac{c_2}{2}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)+\frac{c_3}{3}\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)\right]\right\}\end{aligned} \tag{32c}$$

用谱分解法求解 $E[Z(t)]$，$E[Z^2(t)]$ 到二级小项为止，经过比较繁冗的计算可得：

$$E[Z(t)]\approx E[Z_0(t)]+E[Z_1(t)]+E[Z_2(t)] \tag{33a}$$

$$E[Z^2(t)]\approx E[Z_0^2(t)]+2E[Z_0(t)Z_1(t)]+E[Z_1^2(t)+2Z_0(t)Z_2(t)] \tag{33b}$$

$$\begin{aligned}E[Z(t)]=&\int e^{i\omega t}E[\tilde{Z}(\omega)]d\omega\\\approx&\int e^{i\omega t}E[\tilde{Z}_0(\omega)]d\omega+\int e^{i\omega t}E[\tilde{Z}_1(\omega)]d\omega+\int e^{i\omega t}E[\tilde{Z}_2(\omega)]d\omega\end{aligned} \tag{34a}$$

$$\begin{aligned}E[Z^2(t)]\approx&\int e^{i\omega t}E[\tilde{Z}_0(\omega)*\tilde{Z}_0(\omega)]d\omega+\int e^{i\omega t}E[2\tilde{Z}_0(\omega)*\tilde{Z}_1(\omega)]d\omega\\&+\int e^{i\omega t}E[\tilde{Z}_1(\omega)*\tilde{Z}_1(\omega)+2\tilde{Z}_0(\omega)*\tilde{Z}_2(\omega)]d\omega\end{aligned} \tag{34b}$$

令：
$$F(\omega)=\mathrm{Re}F(\omega)+iI_mF(\omega) \tag{35}$$

$$E[Z_0(t)]=0,\quad E[Z_1(t)]=-\frac{k_2}{k_1}\sigma_{Z_0}^2 \tag{36a, b}$$

$$E[Z_2(t)]=\frac{3k_2k_3}{k_1^2}\sigma_{Z0}^4+\frac{2k_2}{k_1}\sigma_{Z0}^2\int[3k_3\mathrm{Re}G(-\omega_1)+c_3\omega_1 I_m G(-\omega_1)]S_{Z0}(\omega_1)d\omega_1$$

$$+\frac{3k_3}{k_1}\iint[2k_2\mathrm{Re}G(-\omega_2)+c_2\omega_2 I_m G(-\omega_2)]S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_1)d\omega_1 d\omega_2 \tag{36c}$$

$$E[Z_0^2(t)]=\int S_{Z0}(\omega)d\omega=\sigma_{Z0}^2 \tag{37a}$$

$$2E[Z_0(t)Z_1(t)]=-2\sigma_{Z0}^2\int[3k_3\mathrm{Re}G(-\omega_1)+c_3\omega_1 I_m G(-\omega_1)]S_{Z0}(\omega_1)d\omega_1 \tag{37b}$$

$$E[Z_1^2(t)]=\frac{k_2^2}{k_1^2}\sigma_{Z0}^4+\iint\left(2k_2^2+\frac{1}{2}c_2^2\omega_2^2\right)G(\omega_2)G(-\omega_2)S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_1)d\omega_1 d\omega_2$$

$$+\sigma_{Z0}^4\int(9k_3^2+c_3^2\omega_1^2)G(\omega_1)G(-\omega_1)S_{Z0}(\omega_1)d\omega_1$$

$$+\iiint\left(6k_3^2+\frac{2}{3}c_3^2\omega_3^2\right)G(\omega_3)G(-\omega_3)S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_1)S_{Z0}(\omega_3-\omega_2)d\omega_1 d\omega_2 d\omega_3 \tag{37c}$$

$$2E[Z_0(t)Z_2(t)]=\frac{\sigma_{Z0}^2}{k_1}\int[4k_2^2\mathrm{Re}G(\omega_1)-2c_2k_2\omega_1 I_m G(\omega_1)]S_{Z0}(\omega_1)d\omega_1$$

$$+\iint[(8k_2^2-2c_2^2\omega_1(\omega_1-\omega_2))\mathrm{Re}(G(\omega_1)G(\omega_1-\omega_2))+4k_2c_2(\omega_2-2\omega_1)I_m(G(\omega_1)G(\omega_1-\omega_2))]$$

$$\cdot S_{Z0}(\omega_1)S_{Z0}(\omega_2)d\omega_1 d\omega_2+36k_3^2\sigma_{Z0}^2\left[\int\mathrm{Re}G(-\omega_1)\cdot S_{Z0}(\omega_1)d\omega_1\right]^2$$

$$+24c_3k_3\sigma_{Z0}^2\left[\int\mathrm{Re}G(-\omega_1)S_{Z0}(\omega_1)d\omega_1\right]\left[\int\omega_2 I_m G(-\omega_2)S_{Z0}(\omega_2)d\omega_2\right]$$

$$+4c_3^2\sigma_{Z0}^2\left[\int\omega_1 I_m G(-\omega_1)S_{Z0}(\omega_1)d\omega_1\right]^2+\sigma_{Z0}^4\int[(18k_3^2-2c_3^2\omega_1^2)\mathrm{Re}G^2(-\omega_1)$$

$$+12k_3c_3\omega_1 I_m G^2(-\omega_1)]S_{Z0}(\omega_1)d\omega_1+\iiint\{[36k_3^2-4c_3^2\omega_1(\omega_1+\omega_2)]\mathrm{Re}[G(-\omega_1)G(-\omega_1-\omega_2)]$$

$$+12c_3k_3(2\omega_1+\omega_2)I_m[G(-\omega_1)G(-\omega_1-\omega_2)]\}S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_3)S_{Z0}(\omega_3)d\omega_1 d\omega_2 d\omega_3 \tag{37d}$$

当 k_2，c_2，c_3 为零时（28）成为（1）式，则有

$$E[Z_0(t)]=E[Z_1(t)]=E[Z_2(t)]=0 \tag{38}$$

当激励为白噪声时，（37a，b，c）式将和 F．P．方程法的（3）式相同，见附录 A。

（2）多自由度情况

$$m_{ij}\ddot{Z}_j+c_{1ij}\dot{Z}_j+k_{1ij}Z_j+k_{2ijk}Z_jZ_k+k_{3ijkl}Z_jZ_kZ_l$$
$$+c_{2ijk}\dot{Z}_jZ_k+c_{3ijkl}\dot{Z}_jZ_kZ_l=X_i\quad(i=1,2,\cdots,N) \tag{39}$$

式中采用对重复指标求和的 Einstein 约定，且有：

$$k_{2ijk}=k_{2ikj},\quad c_{2ijk}=c_{2ikj} \tag{40a}$$

$$k_{3ijkl}=k_{3ijlk}=k_{3ikjl}=k_{3iklj}=k_{3iljk}=k_{3ilkj} \tag{40b}$$

$$c_{3ijkl}=c_{3ijlk}=c_{3ikjl}=c_{3iklj}=c_{3iljk}=c_{3ilkj} \tag{40c}$$

$$Z_i(t)=Z_{i0}(t)+Z_{i1}(t)+Z_{i2}(t)+\cdots \tag{41}$$

$$m_{ij}\ddot{Z}_{j0}+c_{1ij}\dot{Z}_{j0}+k_{1ij}Z_{j0}=X_i \tag{42a}$$

$$m_{ij}\ddot{Z}_{j1}+c_{1ij}\dot{Z}_{j1}+k_{1ij}Z_{j1}=-k_{2ijk}Z_{j0}Z_{k0}-k_{3ijkl}Z_{j0}Z_{k0}Z_{l0}$$
$$-\frac{1}{2}c_{2ijk}\frac{d}{dt}(Z_{j0}Z_{k0})-\frac{1}{3}c_{3ijkl}\frac{d}{dt}(Z_{j0}Z_{k0}Z_{l0}) \tag{42b}$$

$$m_{ij}+c_{j}\ddot{Z}_{ij}\dot{Z}_{j}+k_{1ij}Z_{j2}=-2k_{2ijk}Z_{j0}Z_{k1}-3k_{3ijkl}Z_{j0}Z_{k0}Z_{l1}$$

$$-c_{2ijk}\frac{d}{dt}(Z_{j0}Z_{k1})-c_{3ijkl}\frac{d}{dt}(Z_{j0}Z_{k0}Z_{l1}) \tag{42c}$$

令：

$$K_{ij}(\omega)=k_{1ij}+i\omega c_{1ij}-\omega^2 m_{ij} \tag{43}$$

设〔$K_{ij}(\omega)$〕的逆矩阵是〔$G_{ij}(\omega)$〕，将（42a，b，c）写成谱分解形式，等式双方乘以$G_{mi}(\omega)$对 i 求和可得：

$$G_{mi}(\omega)K_{ij}(\omega)=\delta_{mj} \tag{44}$$

$$\widetilde{Z}_{m0}(\omega)=G_{mi}(\omega)\widetilde{X}_i(\omega) \tag{45a}$$

$$\widetilde{Z}_{m1}(\omega)=-G_{mi}(\omega)\left(k_{2ijk}+\frac{1}{2}i\omega c_{2ijk}\right)[\widetilde{Z}_{j0}(\omega)*\widetilde{Z}_{k0}(\omega)]$$

$$-G_{mi}(\omega)\left(k_{3ijkl}+\frac{1}{3}i\omega c_{3ijkl}\right)[\widetilde{Z}_{j0}(\omega)*\widetilde{Z}_{k0}(\omega)*\widetilde{Z}_{l0}(\omega)] \tag{45b}$$

$$\widetilde{Z}_{m2}(\omega)=-G_{mi}(\omega)(2k_{2ijk}+i\omega c_{2ijk})[\widetilde{Z}_{j0}(\omega)*\widetilde{Z}_{k1}(\omega)]$$

$$-G_{mi}(\omega)(3k_{3ijkl}+i\omega c_{3ijkl})[\widetilde{Z}_{j0}(\omega)*\widetilde{Z}_{k0}(\omega)*\widetilde{Z}_{l1}(\omega)] \tag{45c}$$

$$E[Z_m(t)]\approx E[Z_{m0}(t)]+E[Z_{m1}(t)]+E[Z_{m2}(t)]$$

$$=\int e^{i\omega t}E[\widetilde{Z}_{m0}(\omega)]d\omega+\int e^{i\omega t}E[\widetilde{Z}_{m1}(\omega)]d\omega+\int e^{i\omega t}E[\widetilde{Z}_{m2}(\omega)]d\omega \tag{46a}$$

$$E[Z_m(t)Z_n(t)]\approx E[Z_{m0}(t)Z_{n0}(t)]+E[Z_{m0}(t)Z_{n1}(t)+Z_{n0}(t)Z_{m1}(t)]$$

$$+E[Z_{m1}(t)Z_{n1}(t)+Z_{m0}(t)Z_{n2}(t)+Z_{n0}(t)Z_{m2}(t)]$$

$$=\int e^{i\omega t}E[\widetilde{Z}_{m0}(\omega)*\widetilde{Z}_{n0}(\omega)]d\omega+\int e^{i\omega t}E[\widetilde{Z}_{m0}(\omega)*\widetilde{Z}_{n}(\omega)+\widetilde{Z}_{n0}(\omega)*\widetilde{Z}_{m1}(\omega)]d\omega$$

$$+\int e^{i\omega t}E[\widetilde{Z}_{m1}(\omega)*\widetilde{Z}_{n1}(\omega)+\widetilde{Z}_{m0}(\omega)*\widetilde{Z}_{n2}(\omega)+\widetilde{Z}_{n0}(\omega)*\widetilde{Z}_{m2}(\omega)]d\omega \tag{46b}$$

在计算时用到如下统计规律：

$$E[\widetilde{Z}_{m0}(\omega)\widetilde{Z}_{m'0}(\omega')]=S_{Z0mm'}(\omega)\delta(\omega+\omega')=S_{Z0m'm}(\omega')\delta(\omega+\omega')$$

$$=S_{Z0m'm}(-\omega)\delta(\omega+\omega') \tag{47a}$$

$$E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)]=E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{j0}(\omega_2)]E[\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)]$$

$$+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{k0}(\omega_3)]E[\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{l0}(\omega_4)]+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{l0}(\omega_4)]E[\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)] \tag{47b}$$

$$E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)\widetilde{Z}_{m0}(\omega_5)\widetilde{Z}_{n0}(\omega_6)]$$

$$=E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{j0}(\omega_2)]E[\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)\widetilde{Z}_{m0}(\omega_5)\widetilde{Z}_{n0}(\omega_6)]$$

$$+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{k0}(\omega_3)]E[\widetilde{Z}_{l0}(\omega_4)\widetilde{Z}_{m0}(\omega_5)\widetilde{Z}_{n0}(\omega_6)\widetilde{Z}_{j0}(\omega_2)]$$

$$+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{l0}(\omega_4)]E[\widetilde{Z}_{m0}(\omega_5)\widetilde{Z}_{n0}(\omega_6)\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)]$$

$$+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{m0}(\omega_5)]E[\widetilde{Z}_{n0}(\omega_6)\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)]$$

$$+E[\widetilde{Z}_{i0}(\omega_1)\widetilde{Z}_{n0}(\omega_6)]E[\widetilde{Z}_{j0}(\omega_2)\widetilde{Z}_{k0}(\omega_3)\widetilde{Z}_{l0}(\omega_4)\widetilde{Z}_{m0}(\omega_5)] \tag{47c}$$

互功率谱密度矩阵元素 $S_{Z0mm'}(\omega)$ 和互相关函数 $R_{Z0jk}(\tau)$ 本身和相互之间有如下关系：

$$S_{Z0mm'}(\omega)=S_{Z0m'm}(-\omega) \tag{48a}$$

$$S_{Z0mm'}(\omega)=S^*_{Z0mm'}(-\omega),\quad S_{Z0mm'}(\omega)=S^*_{Z0m'm}(\omega) \tag{48b, c}$$

$$R_{Z0jk}(\tau)=E[Z_{j0}(t)Z_{k0}(t+\tau)]=\int e^{i\omega\tau}S_{Z0jk}(\omega)d\omega \tag{49a}$$

$$R_{Z0jk}(0)=E[Z_{j0}(t)Z_{k0}(t)]=\int S_{Z0jk}(\omega)d\omega \tag{49b}$$

以上式中右上角的附标*号表示取其复共轭量，经过比较繁冗的计算可得：

$$E[Z_{m0}(t)]=0 \tag{50a}$$

$$E[Z_{m1}(t)]=-k_{2ijk}G_{mi}(0)R_{Z0jk}(0) \tag{50b}$$

$$\begin{aligned}E[Z_{m2}(t)]=&3k_{3ijkl}k_{2l'j'k'}G_{mi}(0)G_{ll'}(0)R_{Z0jj'}(0)R_{Z0kk'}(0)\\&+2k_{2ijk}G_{mi}(0)R_{Z0k'l'}(0)\int G_{kl'}(-\omega_1)(3k_{l'j'k'l'}-i\omega_1c_{3l'j'k'l'})S_{Z0jj'}(\omega_1)d\omega_1\\&+3k_{3ijkl}G_{mi}(0)\iint G_{ll'}(-\omega_2)(2k_{2l'j'k'}-i\omega_2c_{2l'j'k'})S_{Z0jj'}(\omega_1)S_{Z0kk'}(\omega_2-\omega_1)d\omega_1d\omega_2\end{aligned} \tag{50c}$$

$$E[Z_{m0}(t)Z_{n0}(t)]=\int S_{Z0mn}(\omega)d\omega=R_{Z0mn}(0) \tag{51a}$$

$$\begin{aligned}&E[Z_{m0}(t)Z_{n1}(t)+Z_{n0}(t)Z_{m1}(t)]\\&=-R_{Z0kl}(0)\int(3k_{3ijkl}-i\omega_1c_{3ijkl})[G_{ni}(-\omega_1)S_{Z0mj}(\omega_1)+G_{mi}(-\omega_1)S_{Z0nj}(\omega_1)]d\omega_1\end{aligned} \tag{51b}$$

$$\begin{aligned}E[Z_{m1}(t)Z_{n1}(t)]=&G_{mi}(0)G_{ni'}(0)k_{2ijk}k_{2i'j'k'}R_{Z0jk}(0)R_{Z0j'k'}(0)\\&+2\iint G_{mi}(\omega_1)G_{ni'}(-\omega_1)\left(k_{2ijk}k_{2i'j'k'}+\frac{1}{4}\omega_1^2c_{2ijk}c_{2i'j'k'}\right)S_{Z0jj'}(\omega_2)S_{Z0kk'}(\omega_1-\omega_2)d\omega_1d\omega_2\\&+9R_{Z0jk}(0)R_{Z0j'k'}(0)\int G_{mi}(\omega_1)G_{ni'}(-\omega_1)\left(k_{3ijkl}k_{3i'j'k'l'}+\frac{\omega_1^2}{9}c_{3ijkl}c_{3i'j'k'l'}\right)S_{Z0ll'}(\omega_1)d\omega_1\\&+6\iiint G_{mi}(\omega_1)G_{ni'}(-\omega_1)\left(k_{3ijkl}k_{3i'j'k'l'}+\frac{\omega_1^2}{9}c_{3ijkl}c_{3i'j'k'l'}\right)\\&\cdot S_{Z0jj'}(\omega_2)S_{Z0kk'}(\omega_3-\omega_2)S_{Z0ll'}(\omega_1-\omega_3)d\omega_1d\omega_2d\omega_3\end{aligned} \tag{51c}$$

$$\begin{aligned}E[Z_{m0}(t)Z_{n2}(t)]=&k_{2l'j'k'}G_{kl'}(0)R_{Z0j'k'}\int G_{ni}(-\omega_1)(2k_{2ijk}-i\omega_1c_{2ijk})S_{Z0mj}(\omega_1)d\omega_1\\&+2\iint G_{ni}(-\omega_1)(2k_{2ijk}-i\omega_1c_{2ijk})G_{kl'}(-\omega_1-\omega_2)\left[k_{2l'j'k'}-\frac{i}{2}(\omega_1+\omega_2)c_{2l'j'k'}\right]S_{Z0mk'}(\omega_1)\\&\cdot G_{Z0jj'}(\omega_2)d\omega_1d\omega_2+6R_{Z0k'l'}(0)\iint G_{ni}(-\omega_1)(3k_{3ijkl}-i\omega_1c_{3ijkl})G_{ll'}(-\omega_1-\omega_2)\\&\cdot\left[k_{3l'j'k'l'}-\frac{i}{3}(\omega_1+\omega_2)c_{3l'j'k'l'}\right]S_{Z0mj}(\omega_1)S_{Z0kj'}(\omega_1+\omega_2)d\omega_1d\omega_2\\&+6\iint G_{ni}(-\omega_1)(3k_{3ijkl}-i\omega_1c_{3ijkl})G_{ll'}(-\omega_1-\omega_2)\left[k_{3l'j'k'l'}-\frac{i}{3}(\omega_1+\omega_2)c_{3l'j'k'l'}\right]\\&\cdot S_{Z0mj'}(\omega_1)S_{Z0jk'}(\omega_3)S_{Z0kl'}(\omega_2-\omega_3)d\omega_1d\omega_2d\omega_3\\&+3R_{Z0jk}(0)R_{Z0j'k'}(0)\int G_{ni}(-\omega_1)G_{ll'}(-\omega_1)(3k_{3ijkl}-i\omega_1c_{3ijkl})\\&\cdot\left(k_{3l'j'k'l'}-\frac{i}{3}\omega_1c_{3l'j'k'l'}\right)S_{Z0ml'}(\omega_1)d\omega_1\end{aligned} \tag{51d}$$

以上给出了第一种谱分解法的基本公式，它的缺点是随着近似级数增加，计算将大为复杂。

3．第二种谱分解法

（1）单自由度情况

$$m\ddot{Z}+c_1\dot{Z}+k_1Z+\left(k_2+\frac{c_2}{2}\frac{d}{dt}\right)Z^2+\left(k_3+\frac{c_3}{3}\frac{d}{dt}\right)Z^3=X \tag{52}$$

设：

$$E[X]=0 \tag{53}$$

$$E[Z]=\overline{Z} \tag{54}$$

令： $$Z=\overline{Z}+Y \tag{55}$$

$$m\ddot{Y}+c_1\dot{Y}+k_1Y+\left(k_2+\frac{c_2}{2}\frac{d}{dt}\right)(2\overline{Z}Y+Y^2)+\left(k_3+\frac{c_3}{3}\frac{d}{dt}\right)(3\overline{Z}^2Y+3\overline{Z}Y^2+Y^3)=U \tag{56}$$

这里： $$U=X-k_1\overline{Z}-k_2\overline{Z}^2-k_3\overline{Z}^3 \tag{57}$$

假设 Y 是平稳 Gauss 过程，求（56）式的期望值可得

$$(k_2+3k_3\overline{Z})\sigma_Y^2=-k_1\overline{Z}-k_2\overline{Z}^2-k_3\overline{Z}^3 \tag{58}$$

$$m\ddot{Y}+c_1\dot{Y}+k_1Y+\left(m_1+n_1\frac{d}{dt}\right)Y+\left(m_2+n_2\frac{d}{dt}\right)Y^2+\left(m_3+n_3\frac{d}{dt}\right)Y^3=U \tag{59}$$

这里：
$$\left.\begin{aligned}&m_1=2k_2\overline{Z}+3k_3\overline{Z}^2,\quad m_2=k_2+3k_3\overline{Z},\quad m_3=k_3\\&n_1=c_2\overline{Z}+c_3\overline{Z}^2,\quad n_2=\frac{c_2}{2}+c_3\overline{Z},\quad n_3=\frac{c_3}{3}\end{aligned}\right\} \tag{60}$$

将（59）式作谱分解，引入 $G(\omega)$ 及褶积符号可得

$$-m\omega^2+ic_1\omega+k_1=K(\omega),\quad G(\omega)=1/K(\omega) \tag{61}$$

$$\begin{aligned}&\tilde{Y}(\omega)+G(\omega)(m_1+n_1i\omega)\tilde{Y}(\omega)+G(\omega)(m_2+n_2i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)]\\&+G(\omega)(m_3+n_3i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)*\tilde{Y}(\omega)]=G(\omega)\tilde{U}(\omega)\end{aligned} \tag{62}$$

将（62）式自身和自身求褶积可得

$$\begin{aligned}&[1+G(\omega)(m_1+n_1i\omega)]\tilde{Y}(\omega)*[1+G(\omega)(m_1+n_1i\omega)]\tilde{Y}(\omega)\\&+G(\omega)(m_2+n_2i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)]*G(\omega)(m_2+n_2i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)]\\&+G(\omega)(m_3+n_3i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)*\tilde{Y}(\omega)]*G(\omega)(m_3+n_3i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)*\tilde{Y}(\omega)]\\&+2[1+G(\omega)(m_1+n_1i\omega)]\tilde{Y}(\omega)*G(\omega)(m_3+n_3i\omega)[\tilde{Y}(\omega)*\tilde{Y}(\omega)*\tilde{Y}(\omega)]\\&=G(\omega)\tilde{U}(\omega)*G(\omega)\tilde{U}(\omega)\end{aligned} \tag{63}$$

对（63）式求期望值，通过一定的计算后可得

$$\begin{aligned}&[1+G(\omega)(m_1+n_1i\omega)+G(-\omega)(m_1-n_1i\omega)+(m_1^2+n_1^2\omega^2)G(\omega)G(-\omega)]S_Y(\omega)\\&+2(m_2^2+n_2^2\omega^2)G(\omega)G(-\omega)[S_Y(\omega)*S_Y(\omega)]+6\sigma_Y^2[1+G(\omega)(m_1+n_1i\omega)]G(-\omega)(m_3-n_3i\omega)S_Y(\omega)\\&+(m_3^2+n_3^2\omega^2)G(\omega)G(-\omega)[9\sigma_Y^4S_Y(\omega)+6S_Y(\omega)*S_Y(\omega)*S_Y(\omega)]=G(\omega)G(-\omega)S_X(\omega)\end{aligned} \tag{64}$$

这里： $$\int S_Y(\omega)d\omega=\sigma_Y^2 \tag{65}$$

（58），（64），（65）式组成求解的全部公式，以 $\overline{Z}$，σ_Y^2，$S_Y(\omega)$ 作为未知量，可通过适当数值方法求解。

（2）多自由度情况

$$\begin{aligned}&m_{ij}\ddot{Z}_j+c_{1ij}\dot{Z}_j+k_{1ij}Z_j+\left(k_{2ijk}+\frac{1}{2}c_{2ijk}\frac{d}{dt}\right)(Z_jZ_k)\\&+\left(k_{3ijkl}+\frac{1}{3}c_{3ijkl}\frac{d}{dt}\right)(Z_jZ_kZ_l)=X_i,\quad(i=1,\ 2,\cdots,\ N)\end{aligned} \tag{66}$$

设： $$E[X_i]=0,\quad E[Z_j]=\overline{Z}_j \tag{67}$$

令： $$Z_j=Y_j+\overline{Z}_j,\quad Y_j=Z_j-\overline{Z}_j \tag{68}$$

则： $$E[Y_j]=0 \tag{69}$$

$$(m_{ij}\ddot{Y}_j+c_{1ij}\dot{Y}_j+k_{1ij}Y_j)+k_{1ij}\bar{Z}_j+\left(k_{2ijk}+\frac{1}{2}c_{2ijk}\frac{d}{dt}\right)(Y_jY_k+2Y_j\bar{Z}_k+\bar{Z}_j\bar{Z}_k)$$

$$+\left(k_{3ijk}+\frac{1}{3}c_{3ijk}\frac{d}{dt}\right)(Y_jY_kY_l+3Y_jY_k\bar{Z}_l+3Y_j\bar{Z}_k\bar{Z}_l+\bar{Z}_j\bar{Z}_k\bar{Z}_l)=X_i \tag{70}$$

对上式求期望值，注意到

$$E[Y_jY_kY_l]=0,\ E[Y_jY_k]=R_{Yjk}(0) \tag{71}$$

得：

$$R_{Yjk}(0)(k_{2ijk}+3k_{3ijkl}\bar{Z}_l)+k_{1ij}\bar{Z}_j+k_{2ijk}\bar{Z}_j\bar{Z}_k$$
$$+k_{3ijkl}\bar{Z}_j\bar{Z}_k\bar{Z}_l=0,\ (i=1,\ 2,\cdots,\ N) \tag{72}$$

令：

$$-m_{ij}\omega^2+i\omega c_{1ij}+k_{1ij}=K_{ij}(\omega) \tag{73}$$

设矩阵 $[K_{ij}(\omega)]$ 的逆矩阵是 $[G_{ij}(\omega)]$，即有

$$G_{mi}(\omega)K_{ij}(\omega)=\delta_{mj} \tag{74}$$

引入如下简写符号对（70）式作谱分解后可得：

$$H_{mj}(\omega)=G_{mi}(\omega)\left[2\left(k_{2ijk}+\frac{i\omega}{2}c_{2ijk}\right)\bar{Z}_k+3\left(k_{3ijkl}+\frac{i\omega}{3}c_{3ijkl}\right)\bar{Z}_k\bar{Z}_l\right] \tag{75a}$$

$$H_{mjk}(\omega)=G_{mi}(\omega)\left[\left(k_{2ijk}+\frac{i\omega}{2}c_{2ijk}\right)+3\left(k_{3ijkl}+\frac{i\omega}{3}c_{3ijkl}\right)\bar{Z}_l\right] \tag{75b}$$

$$H_{mjkl}(\omega)=G_{mi}(\omega)\left(k_{3ijkl}+\frac{i\omega}{3}c_{3ijkl}\right) \tag{75c}$$

$$\tilde{U}_i(\omega)=\tilde{X}_i(\omega)-(k_{1ij}\bar{Z}_j+k_{2ijk}\bar{Z}_j\bar{Z}_k+k_{3ijkl}\bar{Z}_j\bar{Z}_k\bar{Z}_l)\delta(\omega) \tag{75d}$$

$$\tilde{Y}_m(\omega)+H_{mj}(\omega)\tilde{Y}_j(\omega)+H_{mjk}(\omega)[\tilde{Y}_j(\omega)*\tilde{Y}_k(\omega)]$$
$$+H_{mjkl}(\omega)[\tilde{Y}_j(\omega)*\tilde{Y}_k(\omega)*\tilde{Y}_l(\omega)]=G_{mi}(\omega)\tilde{U}_i(\omega) \tag{76}$$

假设响应是 Gauss 过程，有如下统计规律：

$$E[\tilde{Y}_m(\omega)\tilde{Y}_{m'}(\omega)]=S_{Ymm'}(\omega)\delta(\omega+\omega')=S_{Ym'm}(\omega')\delta(\omega+\omega')=S_{Ym'm}(-\omega)\delta(\omega+\omega') \tag{77a}$$

$$E[\tilde{Y}_i(\omega_1)\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)]=E[\tilde{Y}_i(\omega_1)\tilde{Y}_j(\omega_2)]E[\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)]$$
$$+E[\tilde{Y}_i(\omega_1)\tilde{Y}_k(\omega_3)]E[\tilde{Y}_j(\omega_2)\tilde{Y}_l(\omega_4)]+E[\tilde{Y}_i(\omega_1)\tilde{Y}_l(\omega_4)]E[\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)] \tag{77b}$$

$$E[\tilde{Y}_i(\omega_1)\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)\tilde{Y}_m(\omega_5)\tilde{Y}_n(\omega_6)]$$
$$=E[\tilde{Y}_i(\omega_1)\tilde{Y}_j(\omega_2)]E[\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)\tilde{Y}_m(\omega_5)\tilde{Y}_n(\omega_6)]$$
$$+E[\tilde{Y}_i(\omega_1)\tilde{Y}_k(\omega_3)]E[\tilde{Y}_l(\omega_4)\tilde{Y}_m(\omega_5)\tilde{Y}_n(\omega_6)\tilde{Y}_j(\omega_2)]$$
$$+E[\tilde{Y}_i(\omega_1)\tilde{Y}_l(\omega_4)]E[\tilde{Y}_m(\omega_5)\tilde{Y}_n(\omega_6)\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)]$$
$$+E[\tilde{Y}_i(\omega_1)\tilde{Y}_m(\omega_5)]E[\tilde{Y}_n(\omega_6)\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)]$$
$$+E[\tilde{Y}_i(\omega_1)\tilde{Y}_n(\omega_6)]E[\tilde{Y}_j(\omega_2)\tilde{Y}_k(\omega_3)\tilde{Y}_l(\omega_4)\tilde{Y}_m(\omega_5)] \tag{77c}$$

奇数个谱量乘积的期望值为零，求（76）式自身和自身的褶积，再求期望值，依次可得：

$$[\delta_{mj}+H_{mj}(\omega)]\tilde{Y}_j(\omega)*[\delta_{m'j'}+H_{m'j'}(\omega)]\tilde{Y}_{j'}(\omega)$$
$$+H_{mjk}(\omega)[\tilde{Y}_j(\omega)*\tilde{Y}_k(\omega)]*H_{m'j'k'}(\omega)[\tilde{Y}_{j'}(\omega)*\tilde{Y}_{k'}(\omega)]$$
$$+H_{mjkl}(\omega)[\tilde{Y}_j(\omega)*\tilde{Y}_k(\omega)*\tilde{Y}_l(\omega)]*H_{m'j'k'l'}(\omega)[\tilde{Y}_{j'}(\omega)*\tilde{Y}_{k'}(\omega)*\tilde{Y}_{l'}(\omega)]$$
$$+2[\delta_{mj}+H_{mj}(\omega)]\tilde{Y}_j(\omega)*H_{m'j'k'l'}(\omega)[\tilde{Y}_{j'}(\omega)*\tilde{Y}_{k'}(\omega)*\tilde{Y}_{l'}(\omega)]$$
$$=G_{mi}(\omega)\tilde{U}_i(\omega)*G_{m'i'}(\omega)\tilde{U}_{i'}(\omega) \tag{78}$$

$$
\begin{aligned}
&S_{Ymm'}(\omega)+H_{mj}(\omega)S_{Yjm'}(\omega)+H_{m'j}(-\omega)S_{Ymj}(\omega)+H_{mj}(\omega)H_{m'j'}(-\omega)S_{Yjj'}(\omega)\\
&+2H_{mjk}(\omega)H_{m'j'k'}(-\omega)[S_{Yjj'}(\omega)*S_{Ykk'}(\omega)]\\
&+6R_{Yj'k'}(0)[\delta_{mj}+H_{mj}(\omega)]H_{m'j'k'l'}(-\omega)S_{Yjl'}(\omega)\\
&+H_{mjkl}(\omega)H_{m'j'k'l'}(-\omega)[9R_{Yjk}(0)R_{Yj'k'}(0)S_{Yll'}(\omega)+6S_{Yjj'}(\omega)*S_{Ykk'}(\omega)*S_{Yll'}(\omega)]\\
&=G_{mi}(\omega)G_{m'i'}(-\omega)S_{Yii'}(\omega),\ (m,\ m'=1,\ 2,\cdots,\ N)
\end{aligned}
\tag{79}
$$

这里：
$$R_{Yjk}(0)=\int S_{Yjk}(\omega)d\omega \tag{80}$$

(72)，(79)，(80) 各方程共同组成求解的基本方程，可用适当数值方法求解。两种谱分解法可用来求解各种工程问题。第二种方法的优点是原则上可以通过数值方法求问题的封闭解。

附录 *A* 第一种谱分解法的简例

$$\ddot{Z}(t)+2\beta\dot{Z}(t)+\omega_0^2[Z(t)+aZ^3(t)]=X(t) \tag{A-1}$$

$$G(\omega)=\frac{1}{\omega^2+2\beta i\omega+\omega_0^2}, \tag{A-2}$$

$$E[Z_0^2(t)]=\int S_{Z0}(\omega)d\omega=\sigma_{Z0}^2 \tag{A-2a}$$

$$2E[Z_0(t)Z_1(t)]=-6a\omega_0^2\int G(-\omega)S_{Z0}(\omega)d\omega\cdot\sigma_{Z0}^2 \tag{A-2b}$$

$$
\begin{aligned}
E[Z_1^2(t)]=&9a^2\omega_0^4\sigma_{Z0}^4\int G(\omega)G(-\omega)S_{Z0}(\omega)d\omega\\
&+6a^2\omega_0^4\iiint G(\omega_1)G(-\omega_1)S_{Z0}(\omega_3)S_{Z0}(\omega_2-\omega_3)S_{Z0}(\omega_1-\omega_2)d\omega_1d\omega_2d\omega_3
\end{aligned}
\tag{A-2c}
$$

$$
\begin{aligned}
E[Z_0(t)Z_2(t)]=&18a^2\omega_0^4\sigma_{Z0}^2\left[\int G(-\omega)S_{Z0}(\omega)d\omega\right]^2+9a^2\omega_0^4\sigma_{Z0}^4\int G^2(-\omega)S_{Z0}(\omega)d\omega\\
&+18a^2\omega_0^4\iiint G(-\omega_1)G(-\omega_1-\omega_2)S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_3)S_{Z0}(\omega_3)d\omega_1d\omega_2d\omega_3
\end{aligned}
\tag{A-2d}
$$

设：
$$S_X(\omega)=S_0 \tag{A-3}$$

用留数定理求各积分，略去繁冗计算可得

$$S_{Z0}(\omega)=G(\omega)G(-\omega)S_0 \tag{A-4}$$

$$\sigma_{Z0}^2=\int S_{Z0}(\omega)d\omega=\frac{\pi S_0}{2\beta\omega_0^2} \tag{A-5a}$$

$$\int G(-\omega)S_{Z0}(\omega)d\omega=\frac{\pi S_0}{4\omega_0^4\beta} \tag{A-5b}$$

$$\int G(\omega)G(-\omega)S_{Z0}(\omega)d\omega=\frac{\pi S_0(\omega_0^2+4\beta^2)}{16\omega_0^6\beta^3} \tag{A-5c}$$

$$\int G^2(-\omega)S_{Z0}(\omega)d\omega=\frac{\pi(-\omega_0^2+4\beta^2)S_0}{32\omega_0^6\beta^3} \tag{A—5d}$$

$$
\begin{aligned}
\mathrm{I}=&36a^2\omega_0^4\iiint G(-\omega_1)G(-\omega_1-\omega_2)S_{Z0}(\omega_1)S_{Z0}(\omega_2-\omega_3)S_{Z0}(\omega_3)d\omega_1d\omega_2d\omega_3\\
=&\frac{9a^2\pi^2}{356\beta^3}\ \frac{S_0^3}{\beta^2\omega_0^6(\omega_{cr}^2+4\beta^2)}(-\omega_{cr}^2+2\omega_{cr}^2\beta^2+51\beta^4)
\end{aligned}
\tag{A-6a}
$$

$$\text{II} = 6a^2\omega_0^4 \iiint G(\omega_1)G(-\omega_1)S_{Z0}(\omega_1-\omega_2)S_{Z0}(\omega_2-\omega_3)S_{Z0}(\omega_3)d\omega_1 d\omega_2 d\omega_3$$

$$= \frac{3a^2\pi^2}{256\beta^3} \frac{S_0^3}{\beta^2\omega_0^6(\omega_{cr}^2+4\beta^2)}(3\omega_{cr}^4+58\omega_{cr}^2\beta^2+103\beta^4) \tag{A-6b}$$

$$\text{I}+\text{II} = \frac{3\pi^3 S_0^3}{4\beta^3\omega_0^6} = 6\left(\frac{\pi S_0}{2\beta\omega_0^2}\right)^3 = 6\sigma_{Z0}^3 \tag{A-6c}$$

式中：

$$\omega_{cr}^2 = \omega_0^2 - \beta^2 \tag{A-7}$$

则：

$$2E[Z_0(t)Z_1(t)] = -3a\sigma_{Z0}^4 \tag{A-8a}$$

$$E[Z_1^2(t)+2Z_0(t)Z_2(t)] = 24a^2\sigma_{Z0}^6 \tag{A-8b}$$

和 F. P. 方程法有关展开项相同。

附录 *B* 第二种谱分解法的讨论

仍以（A-1）式作为例题，此时（64）式成为：

$$S_Z(\omega)+6a\omega_0^2\sigma_Z^2 G(-\omega)S_Z(\omega)+a^2\omega_0^4 G(\omega)G(-\omega)[9\sigma_Z^4 S_Z(\omega) + 6S_Z(\omega)*S_Z(\omega)*S_Z(\omega)] = G(\omega)G(-\omega)S_X(\omega) = S_{Z0}(\omega) \tag{B-1}$$

$$\int S_Z(\omega)d\omega = \sigma_Z^2 \tag{B-2}$$

考虑 $a\to 0$，设 X 是白噪声但不是 Gauss 过程，

$$S_X(\omega) = S_0 \tag{B-3}$$

$$S_{Z0}(\omega) = S_0 G(\omega)G(-\omega) \tag{B-4}$$

求 $S_Z(\omega)$ 的一级近似解，再通过积分可得：

$$S_{Z1}(\omega) = S_{Z0}(\omega) - 6a\omega_0^2\sigma_{Z0}^2 G^2(-\omega)G(\omega)S_0 \tag{B-5}$$

$$\sigma_{Z0}^2 = \int S_{Z0}(\omega)d\omega = \frac{\pi S_0}{2\beta\omega_0^2} \tag{B-6}$$

$$\int G^2(-\omega)G(\omega)d\omega = \frac{\pi}{4\beta\omega_0^4} \tag{B-7}$$

$$\sigma_{Z1}^2 = \int S_{Z1}(\omega)d\omega = \sigma_{Z0}^2 - 3a\sigma_{Z0}^2\frac{\pi}{2\beta\omega_0^2} = \sigma_{Z0}^2(1-3a\sigma_{Z0}^2) \tag{B-8}$$

和以上方法都相同。考虑 $a\to\infty$，积分（B-1）式，等式左方仅保留 a^2 的系数项，可得

$$\lim_{a\to\infty}\int a^2\omega_0^4 G(\omega)G(-\omega)[9\sigma_Z^4 S_Z(\omega)+6S_Z(\omega)*S_Z(\omega)*S_Z(\omega)]d\omega = \sigma_{Z0}^2 \tag{B-9}$$

存在一个平均值 $G(\omega_m)G(-\omega_m)$ 使得：

$$\omega_0^4 G(\omega_m)G(-\omega_m) = M \tag{B-10}$$

$$a^2 M\int[9\sigma_Z^4 S_Z(\omega)+6S_Z(\omega)*S_Z(\omega)*S_Z(\omega)]d\omega = 15a^2 M\sigma_Z^6 = \sigma_{Z0}^2$$

$$\sigma_Z^2 = 0.4055M^{-1/3}\sigma_{Z0}^{2/3}a^{-2/3} \tag{B-11}$$

σ_Z^2 总是随着 $a^{-2/3}$ 趋向于零。

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Spectral Resolving Methods for Analyzing Nonlinear Random Vibrations (A Comparision With Cnrrent Methods)

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Abstract

Current method: Bokker-Planck equation method, equivalent linearization method and perturbation method can be applied only in a quite narrow district of problems and it is lack of efficient numerical methods for practical calculation. For the sake of overcoming these shortages, according to two different kinds of statistical rule: excitation (or response) is supposed to be stationary, Gaussian process, and response (or excitation) is supposed to be stationary but not Gaussian process, two kinds of spectral resolving methods for analyzing nonlinear problems are suggested respectively, by combining these two methods suggested with spectral resolving method of structural dynamics applied in F.E.M., a quite wide district of nonlinear random vibration problems can be solved.

In this paper, formulation of new methods has been derived, a brief discussion has been made for their foundamental behavior and convergence, a comparision between results obtained by new methods and by current methods has been worked. On the other hand, in this paper, the author suggests a new definition of structural random stability and its criterion has been written out.