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具有缓变狭窄直圆管内的脉动流——早期动脉粥样硬化区的血流动力学机制讨论

许世雄
（复旦大学应用力学系）

提要 六十年代末期以来 D. F. Young 等人[1-5]提出了各种数学力学模型对动脉狭窄区血液流动进行理论分析，寻求血流动力学诸因素的定量关系，力图为动脉粥样硬化的发展提供力学上的解释根据．动脉粥样硬化易发区通常在主动脉的较大分枝处及脑动脉和冠状动脉内，血流是脉动的，Reynolds 数是 $O(1)$ 或以上的量级．但 D.F. Young 等人的研究多数考虑定常流或小 Reynolds 数的流动，所得结果对动脉粥样硬化狭窄区的血流并不适合．本文考虑 Reynolds 数为 $O(1)$ 的量级的冠状动脉狭窄区的脉动流，初步探讨早期动脉粥样硬化狭窄区的血流动力学机制．

一、力学模型及基本方程

所考虑的力学模型为：

i) "类环状"狭窄，即把病变血管看作具有轴对称狭窄的刚性直圆管．设管道曲面为(见图1)

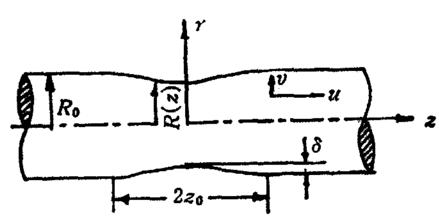

图 1

$$R(z)=\begin{cases}R_0-\delta f(z) & |z|\leqslant z_0\\ R_0 & |z|>z_0\end{cases}\tag{1.1}$$

式中 $f(z)\in c^1$，$f(z)>0$ 且 $f_{max}(z)=1$，$f(-z_0)=f(z_0)=0$，$f'(\pm z_0)=0$

ii) 本文考虑初起硬化，最大隆起高度 δ 是小量，$\delta\ll R_0$，$\delta\ll z_0$，同时考虑 $\varepsilon=R_0/z_0<1$ 的情况；

iii) 冠状动脉处血流是脉动的层流，$1<\alpha<15$，$\mathrm{Re}=U_0R_0/\nu\sim O(1)$，$\alpha=R_0\sqrt{\omega/\nu}$，是表征脉动的 Womersley 数．根据文献[6]，血液按牛顿流体处理．

根据上述模型，注意到 δ 是小量，控制血流方程是

$$\left.\begin{aligned}&\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rv)=0\\&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)\\&\frac{\partial p}{\partial r}\end{aligned}\right\}\tag{1.2}$$

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边界条件是

$$u|_{r=R}=v|_{r=R}=0,\quad v\Big|_{r=0}=\frac{\partial u}{\partial r}\Big|_{r=0}=0,$$

$$Q=\int_0^R 2\pi r\cdot u dr=Q_0(1+\cos\omega t) \tag{1.3}$$

式中 $Q_0=\pi R_0^2U_0$，U_0 是狭窄前管内平均流速，ω 是振荡分量的圆频率．边界上的取值，同样对 δ 展开取一级近似形式：

$$F|_R=F|_{R_0-\delta f}\doteq F|_{R_0}-\delta F'|_{R_0} \tag{1.4}$$

（除了可求得精确解的情况，边界取值均如此）引进无量纲参量

$$\bar z=z/z_0\sim O(1),\ \bar r=r/R_0\sim O(1),\ \bar t=\omega t\sim O(1)$$
$$\bar Q=Q/Q_0\sim O(1),\ \bar u=u/U_0\sim O(1),\ \bar v=v/U_0\sim O(\varepsilon)$$
$$\bar p=p/P\sim O(1),\ \bar\delta=\delta/R_0,\ \bar R=R/R_0=1-\delta f$$

从连续性方程很易看到 $\bar v$ 具有 $O(\varepsilon)$ 量级．从动量方程可以看到只要取 $P=\rho U_0\dfrac{z_0}{R_0^2}\nu$ 就可使 $\bar p$ 具有 $O(1)$ 的量级．把它们代入方程（1.2）使之无量纲化，去掉字母上"一"的记号后成为

$$\left.\begin{aligned}&\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rv)=0\\&\alpha^2\frac{\partial u}{\partial t}+\varepsilon\mathrm{Re}\left(u\frac{\partial u}{\partial z}+\frac{1}{\varepsilon}v\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial z}+\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)\\&\frac{\partial p}{\partial r}=0\end{aligned}\right\} \tag{1.5}$$

我们采用正则摄动法求解方程．设解为几何参数 $\varepsilon(<1)$ 的幂级数形式

$$u=u_0+\varepsilon u_1+\cdots,\quad v=\varepsilon v_0+\cdots,\quad p=p_0+\varepsilon p_1+\cdots \tag{1.6}$$

代入运动方程，比较 ε 的各幂次系数后得到

$$\begin{aligned}&\varepsilon^0:\ \alpha^2\frac{\partial u_0}{\partial t}=-\frac{\partial p_0}{\partial z}+\left(\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}\right)\\&\varepsilon^1:\ \alpha^2\frac{\partial u_1}{\partial t}+\frac{\partial p_1}{\partial z}-\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{1}{r}\frac{\partial u_1}{\partial r}\right)=-\mathrm{Re}\left(u_0\frac{\partial u_0}{\partial z}+v_0\frac{\partial u_0}{\partial r}\right)\\&\cdots\cdots\end{aligned} \tag{1.7}$$

本文取截止至 ε^1 的近似解．

二、方程求解

相应 ε^0 的方程是

$$\left\{\begin{aligned}&\alpha^2\frac{\partial u_0}{\partial t}=-\frac{\partial p_0}{\partial z}+\left(\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}\right)\\&\frac{\partial p_0}{\partial r}=0\end{aligned}\right. \tag{2.1}$$

引入无量纲流函数 ψ_0，由于方程是线性的且有边界条件

$$\phi_0|_{r=R} = \text{Real}\left[-\frac{1}{2}(1+e^{it})\right],$$

很容易求出精确解

$$\begin{cases} u_0 = u_{0(s)} + u_{0(0)} \\ u_{0(s)} = \dfrac{2}{R^2}\left[1-\left(\dfrac{r}{R}\right)^2\right] \\ u_{0(0)} = \text{Real}\left\{\dfrac{J_0(\sqrt{-i}\alpha R)}{R^2 J_2(\sqrt{-i}\alpha R)}\left[\dfrac{J_0(\sqrt{-i}\,\alpha r)}{J_0(\sqrt{-i}\,\alpha R)}-1\right]e^{it}\right\} \end{cases} \tag{2.2}$$

$$\begin{cases} v_0 = v_{0(s)} + v_{0(0)} \\ v_{0(s)} = 2rR'\left[1-\left(\dfrac{r}{R}\right)^2\right]\Big/R^3 \\ v_{0(0)} = \text{Real}\left\{\dfrac{R'J_1^2(\sqrt{-i}\,\alpha R)}{R^2 J_2^2(\sqrt{-i}\,\alpha R)}\left[\dfrac{J_1(\sqrt{-i}\,\alpha r)}{J_1(\sqrt{-i}\,\alpha R)}-\dfrac{r}{R}\right]e^{it}\right\} \end{cases} \tag{2.3}$$

此解是方程(1.5)略去非线性对流惯性项的结果，由于 $Re \sim O(1)$，只能适用于 $\varepsilon \ll 1$ 情况，即狭窄退化的情况．这实际上就是 Womersley[7] 所得的不定常 Poiseuille 流的结果．对于动脉粥样硬化区的流动必须进一步考虑非线性惯性效应

相应 ε^1 的方程是

$$\begin{cases} \alpha^2\dfrac{\partial u_1}{\partial t}+\dfrac{\partial p_1}{\partial z}-\left(\dfrac{\partial^2 u_1}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial u_1}{\partial r}\right) = -Re\left(u_0\dfrac{\partial u_0}{\partial z}+v_0\dfrac{\partial u_0}{\partial r}\right) \\ \dfrac{\partial p_1}{\partial r} = 0 \end{cases} \tag{2.2}$$

边界条件是

$$u_1\Big|_{r=R} = \frac{\partial u_1}{\partial r}\Big|_{r=0} = 0, \qquad Q_1 = \int_0^R 2ru_1 dr = 0 \tag{2.3}$$

方程 (2.2) 第一式右端是零级近似对一级近似的贡献，可把它分成“定常”、“振荡”和“定常-振荡”三部分，同样分别以下标“s”、“0”及“s-0”记

$$u_0\frac{\partial u_0}{\partial z}+v_0\frac{\partial u_0}{\partial r} = g_{(s)}+g_{(0)}+g_{(s-0)}$$

$$\left.\begin{aligned} g_{(s)} &= u_{0(s)}\frac{\partial}{\partial z}u_{0(s)}+v_{0(s)}\frac{\partial}{\partial r}u_{0(s)} \\ g_{(0)} &= u_{0(0)}\frac{\partial}{\partial z}u_{0(0)}+v_{0(0)}\frac{\partial}{\partial r}u_{0(0)} \\ g_{(s-0)} &= u_{0(0)}\frac{\partial}{\partial z}u_{0(s)}+u_{0(s)}\frac{\partial}{\partial z}u_{0(0)}+v_{0(s)}\frac{\partial}{\partial r}u_{0(s)}+v_{0(0)}\frac{\partial}{\partial r}u_{0(s)} \end{aligned}\right\} \tag{2.4}$$

由于方程线性，边界条件齐次，可以分别就这三个贡献求解

i) $g_{(s)}$ 的贡献．它与时间无关，称为第一定常效应，相应解 $u_{1(s)}$ 应与时间无关，为

$$u_{1(s)} = \frac{Re\cdot R'}{R^3}\left(\frac{2}{9}-\frac{r^2}{R^2}+\frac{r^4}{R^4}+\frac{2}{9}\frac{r^6}{R^6}\right). \tag{2.5}$$

ii) $g_{(0)}$ 的贡献．

记 $J_K(x) = M_K e^{i\theta_K}$,

$$\left[\frac{J_0(\sqrt{-i}\,\alpha r)}{J_0(\sqrt{-i}\,\alpha R)}\right]-1=M_A e^{i\theta_A},\quad \left[\frac{J_1(\sqrt{-i}\,\alpha r)}{J_1(\sqrt{-i}\,\alpha R)}\right]-\frac{r}{R}=M_B\cdot e^{i\theta_B}$$

$$\frac{J_0(\sqrt{-i}\,\alpha R)}{R^2J_2(\sqrt{-i}\,\alpha R)}\left[\frac{J_0(\sqrt{-i}\,\alpha r)}{J_0(\sqrt{-i}\,\alpha R)}-1\right]=M_u e^{i\theta_u},$$

$$\frac{R'J_1^2(\sqrt{-i}\,\alpha R)}{R^2J_2^2(\sqrt{-i}\,\alpha R)}\left[\frac{J_1(\sqrt{-i}\,\alpha r)}{J_1(\sqrt{-i}\,\alpha R)}-\frac{r}{R}\right]=M_V\cdot e^{i\theta_V}$$

则有

$$\begin{aligned}
g_{(0)}=&\left[\frac{1}{2}M_u\frac{\partial M_u}{\partial z}+\frac{1}{2}M_V\frac{\partial M_u}{\partial r}\cos(\theta_u-\theta_V)\right.\\
&\left.-\frac{1}{2}M_VM_u\frac{\partial\theta_u}{\partial r}\sin(\theta_u-\theta_V)\right]\\
&+\left[\frac{1}{2}M_u\frac{\partial M_u}{\partial z}\cos(2\theta_u+2t)-\frac{1}{2}M_u^2\frac{\partial\theta_u}{\partial z}\sin(2\theta_u\right.\\
&+2t)+\frac{1}{2}M_V\frac{\partial M_u}{\partial r}\cos(\theta_u+\theta_V+2t)\\
&\left.-\frac{1}{2}M_u\cdot M_V\frac{\partial\theta_u}{\partial r}\sin(\theta_u+\theta_V+2t)\right]\\
=&\,g_{(0)-(s)}+g_{(0)-(0)}
\end{aligned}\tag{2.6}$$

$g_{(0)-(s)}$ 是 $g_{(0)}$ 中定常效应部分，称为第二定常效应。

iii）$g_{(s-0)}$ 的贡献．用上面的记法可表示为

$$\begin{aligned}
g_{(s-0)}=&\left\{\frac{2}{R^2}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial M_u}{\partial z}-\frac{4R'}{R^3}\left[1-2\left(\frac{r}{R}\right)^2\right]M_u\right.\\
&\left.+2r\frac{R'}{R^3}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial M_u}{\partial r}\right\}\cdot\cos(\theta_u+t)\\
&+\left\{-\frac{2}{R^2}\left[1-\left(\frac{r}{R}\right)^2\right]M_u\frac{\partial\theta_u}{\partial z}\right.\\
&\left.-2r\frac{R'}{R^3}\left[1-\left(\frac{r}{R}\right)^2\right]M_u\frac{\partial\theta_u}{\partial r}\right\}\cdot\sin(\theta_u+t)\\
&-\frac{4r}{R^4}M_V\cos(\theta_V+t)
\end{aligned}\tag{2.7}$$

对于 $g_{(0)}$ 及 $g_{(s-0)}$ 的贡献所相应的解用数值方法来求。本文采用等步长网格，时间偏导用前差式，径向偏导用中心差式，方程离散用 Crank-Nicolson 时间相关格式。

关于流动分离，Prandtl 判据已不再适用，本文采用 Despard-Miller 的判据[8]，在所有时间

$$\left.\frac{\partial u}{\partial r}\right|_{r=R}\geqslant 0\tag{2.8}$$

此判据生理意义是，流动分离时动脉管壁上受到的虽是脉动的但始终是负向的切应力。

三、 计算实例及讨论

采用余弦型狭窄模型计算

$$f(z)=\begin{cases}\frac{1}{2}(1+\cos\pi z) & |z|\leqslant 1\\ 0 & |z|>1\end{cases}$$

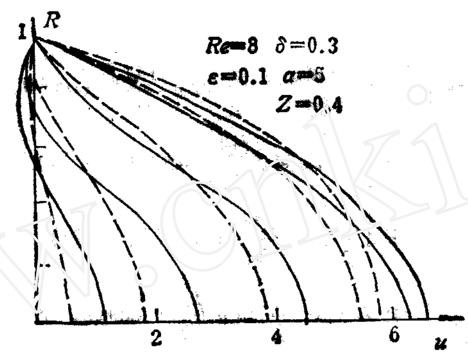

图 2

----u_0 ——$u_0+\varepsilon u_1$

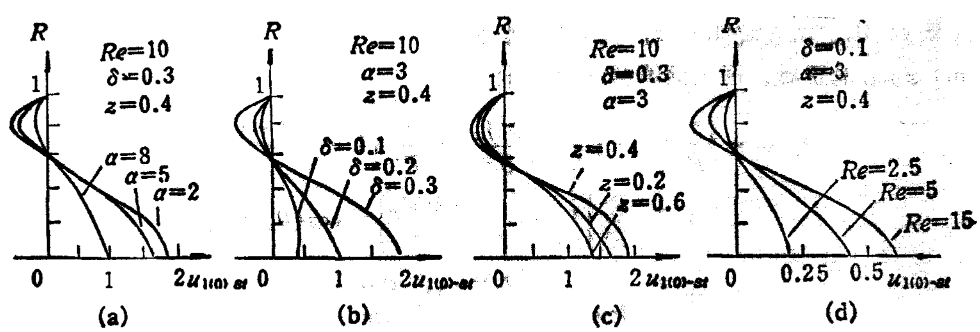

图 3

图 2 是同一参数条件下 $u=u_0$ 及 $u=u_0+\varepsilon u_1$ 的分布，比较后可以看到后者在近轴处被“推进”而在近壁处被“拉后”，改变其它参数的计算表明上述现象依然存在，这反映在 u_1 中存在一种稳定的因素．

图 3 (a) (b) (c) (d) 是第二定常效应 $g_{(0)-(s)}$ 对速度 u_1 的贡献 $u_{1(0)-st}$，它具有如下特点：

i) 近轴处正向分布，近壁处负向分布，从而表现为图 2 中出现的对 u 的作用．正是这种作用加速了流动分离；

ii) 这种作用随狭窄严重而增强，随 Reynolds 数加大而增强；

iii) Womersley 数 α 的变化对这种作用影响较复杂，图 4 反映了这一情况．

对于人的左冠状动脉粥样硬化狭窄模型进行了计算．图 5 是壁切应力随时间变化的

规律．可以发现壁切应力均为负值，流动分离．还计算了狭窄区内各处按时间平均的壁切应力 $\tilde{\tau}_w$ 值．不论在分离区内或分离区外，$\tilde{\tau}_w$ 及 τ_w 均较低，幅值变化不大，这与 Caro 的低切应力说相符．

z	−1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1
$\tilde{\tau}_w$(dyn/cm²)	1.44	4.43	8.80	14.26	15.23	4.86	−6.64	−8.24	−4.64	−1.26	1.44

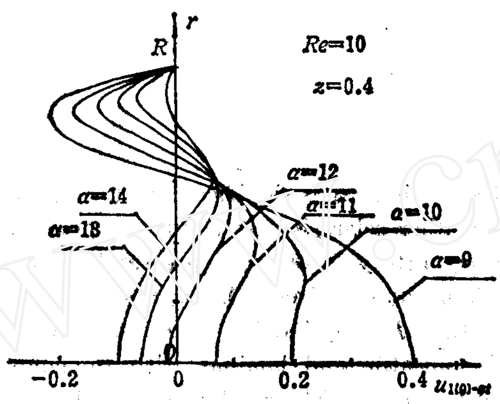

图 4

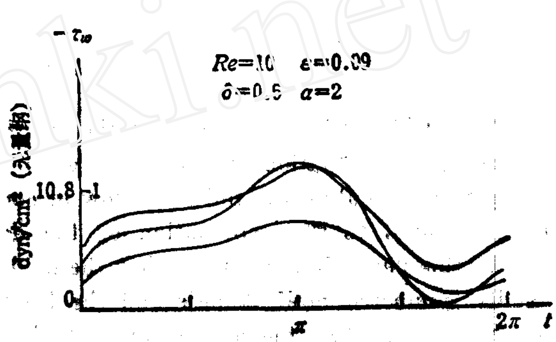

图 5

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THE PULSATILE FLUID FLOW IN A CYLINDRICAL TUBE WITH A MILD STENOSIS——THE HEMODYNAMIC THEORY OF THE INITIAL ATHEROSCLEROTIC DEVELOPMENT

Xu Shixiong
(*Fudan University, Shanghai*)

Abstract

The study of hemodynamics at the arterial stenosis area is significant in the examination of the formation and development of the atherosclerosis. It is the purpose of this paper to analyse the pulsatile flow in coronary artery in order to present hemodynamic ground to the explanation of the development of initial atherogenosis.

In this paper it is assumed that the model studied is cylindrical regid tube with axially symmetric mild stenosis and the flow is pulsatile and laminar with Reynolds number O(1). And the method of perturbation is applied. The results calculated from a cosine stenosis example show that, in the nonlinear convective inertia contribution of a pure oscillating flow, there is a time-independent steady effect which plays an important part in the flow separation. Finally, the blood flow through the initial left-coronary atherosclerotic stenosis region was calculated. It is found that the time-averaged wall shear stress values in the whole stenosis region are low. It is in agreement with Caro's low shear stress theory.