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关于开孔薄板大挠度问题的一般数学理论

朱正佑　　程昌钧

（兰 州 大 学）

提要　众所周知，应力函数 F 的多值性和位移的单值性要求使得开孔薄板的大挠度问题，在理论上和数值计算上都变得比较困难。本文求得了使应力函数成为单值的一个充要条件，并证明了位移单值性要求等价于一些约束方程 $L_i(F)-N_i(w)=0$，其中 L_i 是 F 的线性泛函，$N_i(w)$ 是挠度的齐二次泛函．利用这些结论，求得了开孔薄板大挠度问题的控制方程．同时给出了计算这一问题的数值方法；讨论了开孔薄板的临界载荷问题，当这些临界载荷是"单重的"时，给出了从平凡解处分叉出去的屈曲解的迭代计算方案．

一、前　　言

众所周知，对于各向同性薄板大挠度问题，Von Kármán 已提出了著名的控制方程

$$\left.\begin{aligned}\Delta^2 w &= \frac{q}{D}+\frac{h}{D}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}\right]\\ \Delta^2 F &= E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right]\end{aligned}\right\}\tag{1.1}$$

其中

$$\Delta^2\equiv\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\equiv\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\tag{1.2}$$

$w(x,y)$ 是板中面的挠度，$F(x,y)$ 是 Airy 应力函数．当板不开孔时，(1.1)中的应力函数 F 是单值的，因而在适当的边界条件下由(1.1)可以完全确定薄板的挠度和应力状态．许多作者已讨论了求解这种问题的各种解析方法和数值方法[1,2]．然而当薄板内部开有若干个孔时，一般说来，应力函数 F 将是一个多值函数，这样就会给问题的解析求解和数值计算带来了严重的困难．直至目前，除少数特殊情形（环形板）[3]外，尚未见到一般讨论．

然而，对线弹性理论的多连域平面问题，Мусхелишвили[4] 利用解析函数的工具已成功地讨论了应力函数的多值性．本文将利用[4]中讨论多值函数的基本观点来研究开孔正交各向异性薄板大挠度问题中应力函数的多值性问题．在第2节中，我们详细推导了单值的位移和应力函数所应满足的充分必要条件．第3节将主要致力于建立开孔正交各向异性薄板大挠度问题的"单值化"模型及其计算方法；第4节将利用这种模型来讨论屈曲板的临界载荷的计算方法以及临界载荷附近板的屈曲状态．

二、开孔薄板大挠度问题的单值性条件

现在，我们来考察厚度为 h 的开孔正交各向异性薄板在横向分布力 q，面内体力 $(f_x,$

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f_y, 0)和面内边界外力作用下的平衡问题. 变形前板中面所在的位置是 x-y 平面上的有界多连域 Ω. 设 Ω 的边界 $\partial\Omega$ 由 $m+1$ 条闭曲线 $\Gamma_0, \Gamma_1, \cdots, \Gamma_m$ 组成, $\Gamma_i (i=1, 2, \cdots, m)$ 被包含在 Γ_0 的内部, 并设 Γ_i 彼此不相交 (如图 1 所示).

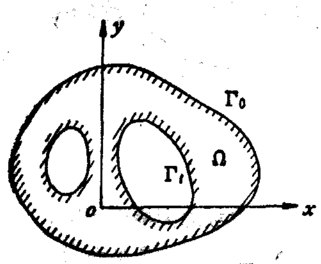

图 1

设板中面各点沿坐标轴方向的位移为 u, v, w, 变形为 $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, $\bar{\gamma}_{xy}$, 应力分量 $\sigma_x, \sigma_y, \tau_{xy}$ 沿厚度方向的积分记为

$$N_x = \int_{-\frac{h}{2}}^{h/2} \sigma_x dz, \quad N_y = \int_{-\frac{h}{2}}^{h/2} \sigma_y dz, \quad S_{xy} = \int_{-\frac{h}{2}}^{h/2} \tau_{xy} dz.$$

则在 Ω 内这些函数应满足如下控制方程.

(a) 平衡方程

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = q + N_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \qquad (2.1)_a$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial S_{xy}}{\partial y} + hf_x = 0, \quad \frac{\partial S_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + hf_y = 0 \qquad (2.1)_{b,c}$$

式中 f_x, f_y 是面内体力沿 x, y 轴方向的投影.

(b) 位移与变形的关系

$$\bar{\varepsilon}_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \bar{\varepsilon}_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$\bar{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad (2.2)$$

(c) 本构关系

$$\bar{\varepsilon}_x = \frac{1}{E_1 h}(N_x - \nu_1 N_y), \quad \bar{\varepsilon}_y = \frac{1}{E_2 h}(N_y - \nu_2 N_x)$$
$$\bar{\gamma}_{xy} = \frac{1}{Gh} S_{xy} \qquad (2.3)$$

其中 $E_1, E_2, \nu_1, \nu_2, G$ 是正交各向异性板的材料常数, 满足 $\nu_1/E_1 = \nu_2/E_2$. $D_i = E_i h^3/12(1-\nu_1\nu_2) (i=1,2)$, $D_k = Gh^3/12, D_3 = D_1\nu_2 + 2D_k = D_2\nu_1 + 2D_k$.

由(2.2)消去 u, v, 得到协调方程

$$\frac{\partial^2 \bar{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial x^2} - \frac{\partial^2 \bar{\gamma}_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \qquad (2.4)$$

利用(2.3), 从上式中消去 $\bar{\varepsilon}_x, \bar{\varepsilon}_y$ 及 $\bar{\gamma}_{xy}$ 得到

$$\frac{1}{E_2}\frac{\partial^2 N_y}{\partial x^2} + \left(\frac{1}{2G} - \frac{\nu_1}{E_1}\right)\frac{\partial^2 N_x}{\partial x^2} + \left(\frac{1}{2G} - \frac{\nu_2}{E_2}\right)\frac{\partial^2 N_y}{\partial y^2} + \frac{1}{E_1}\frac{\partial^2 N_x}{\partial y^2} = h\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{2G}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)\right] \qquad (2.5)$$

现在, 我们的基本问题是在一定的边界条件下, 由(2.1)和(2.5)求解 w, N_x, N_y, S_{xy}.

1. u、v 的单值性条件

当 Ω 是单连通区域时，一旦由(2.1)、(2.5)确定了 w, N_x, N_y, S_{xy}，则由(2.2)和(2.3)就可确定单值的位移 u 和 v。但对多连通域却不是这样。下面，我们给出存在单值位移 u，v 的一个充分必要条件。

若单值的 u 和 v 存在，则由(2.2)知

$$
\begin{aligned}
\hat{\varepsilon}_x &\equiv \frac{\partial u}{\partial x} = \varepsilon_x - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
\hat{\varepsilon}_y &\equiv \frac{\partial v}{\partial y} = \varepsilon_y - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\hat{\gamma}_{xy} &\equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \\
\hat{\gamma} &\equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\end{aligned}
\tag{2.6}
$$

都应是单值的．由(2.6)容易求得

$$
\frac{\partial \hat{\gamma}}{\partial x} = \frac{\partial \gamma_{xy}}{\partial x} - 2\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}
$$

$$
\frac{\partial \hat{\gamma}}{\partial y} = 2\frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}
$$

由 (2.4)知，$\frac{\partial}{\partial y}\left(\frac{\partial \hat{\gamma}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \hat{\gamma}}{\partial y}\right)$. 因为 $\hat{\gamma}$ 是单值的，所以应有

$$
\oint_{\Gamma_i} \frac{\partial \hat{\gamma}}{\partial x} dx + \frac{\partial \hat{\gamma}}{\partial y} dy = 0 \qquad (i = 1, 2, \cdots, m)
$$

将(2.6)代入，则得到 ε_x、ε_y、γ_{xy} 及 w 之间应满足

$$
\begin{aligned}
\oint_{\Gamma_i} &\left(\frac{\partial \gamma_{xy}}{\partial x} - 2\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) dx \\
&+ \left(2\frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) dy = 0
\end{aligned}
\tag{2.7}
$$

$$(i = 1, 2, \cdots, m)$$

当(2.7)成立时，$\hat{\gamma}(x, y)$ 有如下表达式

$$
\begin{aligned}
\hat{\gamma}(x, y) = \hat{\gamma}_0 + \int_{(x_0, y_0)}^{(x, y)} &\left(\frac{\partial \gamma_{xy}}{\partial x} - 2\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) dx \\
&+ \left(2\frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) dy
\end{aligned}
$$

其中 $\hat{\gamma}_0 = \hat{\gamma}(x_0, y_0)$，$(x_0, y_0)$ 是 $\bar{\Omega}$ 中的任一给定点．积分路径是沿联结 (x_0, y_0) 和 (x, y) 之间的金属于 $\bar{\Omega}$ 的任一光滑曲线．

现在设(2.7)成立，由

$$
\frac{\partial u}{\partial x} = \varepsilon_x, \quad \frac{\partial u}{\partial y} = \frac{1}{2}(\hat{\gamma}_{xy} - \hat{\gamma})
$$

及(2.7)知，

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial \hat{\varepsilon}_x}{\partial y}=\frac{\partial}{\partial x}\left[\frac{1}{2}(\hat{\gamma}_{xy}-\hat{r})\right]=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

于是，由 u 的单值性知必定有

$$\oint_{\Gamma_i}\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0 \qquad (i=1,2,\cdots,m)$$

亦即

$$\oint_{\Gamma_i}\hat{\varepsilon}_x dx+\frac{1}{2}(\hat{\gamma}_{xy}-\hat{r})dy=0 \qquad (i=1,2,\cdots,m)$$

由(2.6)式，上式可用 $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, $\bar{\gamma}_{xy}$ 及 w 表示成

$$\begin{aligned}\oint_{\Gamma_i}&\left[\bar{\varepsilon}_x-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}y\left(\frac{\partial \bar{\gamma}_{xy}}{\partial x}-2\frac{\partial \bar{\varepsilon}_x}{\partial y}-\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right)\right]dx\\&+\frac{1}{2}\left[\bar{\gamma}_{xy}-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+y\left(2\frac{\partial \bar{\varepsilon}_y}{\partial x}-\frac{\partial \bar{\gamma}_{xy}}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)\right]dy\\&=0 \qquad (i=1,2,\cdots,m)\end{aligned} \tag{2.8}$$

类似地，由 v 的单值性得到 $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, $\bar{\gamma}_{xy}$ 及 w 应满足

$$\begin{aligned}\oint_{\Gamma_i}&\frac{1}{2}\left[\bar{\gamma}_{xy}-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-x\left(\frac{\partial \bar{\gamma}_{xy}}{\partial x}-2\frac{\partial \bar{\varepsilon}_x}{\partial y}-\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right)\right]dx\\&+\left[\bar{\varepsilon}_y-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{1}{2}x\left(2\frac{\partial \bar{\varepsilon}_y}{\partial x}-\frac{\partial \bar{\gamma}_{xy}}{\partial y}\right.\right.\\&\left.\left.+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)\right]dy=0 \qquad (i=1,2,\cdots,m)\end{aligned} \tag{2.9}$$

这样，我们证明了当单值的 u 和 v 存在时，$\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, $\bar{\gamma}_{xy}$ 和 w 之间必满足 (2.7)—(2.9) 式．反之，由上面的推导过程不难看出，当 (2.7)—(2.9) 成立时，满足(2.2)的单值位移 u 和 v 可由下式给定:

$$\begin{aligned}u(x,y)&=u_0+\frac{1}{2}r_0(y-y_0)+\int_{(x_0,y_0)}^{(x,y)}\hat{u}_1 d\xi+\hat{u}_2 d\eta\\v(x,y)&=v_0+\frac{1}{2}r_0(x-x_0)+\int_{(x_0,y_0)}^{(x,y)}\hat{v}_1 d\xi+\hat{v}_2 d\eta\end{aligned} \tag{2.10}$$

其中

$$\begin{aligned}\hat{u}_1(\xi,\eta)=\bar{\varepsilon}_x-\frac{1}{2}\left(\frac{\partial w}{\partial \xi}\right)^2-\frac{1}{2}(y-\eta)&\\\cdot\left(\frac{\partial \bar{\gamma}_{xy}}{\partial \xi}-2\frac{\partial \bar{\varepsilon}_x}{\partial \eta}-\frac{\partial^2 w}{\partial \xi^2}\frac{\partial w}{\partial \eta}+\frac{\partial w\partial^2 w}{\partial \xi\partial \xi\partial \eta}\right)&\\\hat{u}_2(\xi,\eta)=\frac{1}{2}\left[\bar{\gamma}_{xy}-\frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta}-(y-\eta)\right.&\\\left.\cdot\left(2\frac{\partial \bar{\varepsilon}_y}{\partial \xi}-\frac{\partial \bar{\gamma}_{xy}}{\partial \eta}+\frac{\partial w}{\partial \xi}\frac{\partial^2 w}{\partial \eta^2}-\frac{\partial w}{\partial \eta}\frac{\partial^2 w}{\partial \xi\partial \eta}\right)\right]&\end{aligned}$$

$$\phi_1(\xi,\eta)=\frac{1}{2}\left[\gamma_{xy}-\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}+(x-\xi)\cdot\left(\frac{\partial\gamma_{xy}}{\partial\xi}-2\frac{\partial\varepsilon_x}{\partial\eta}-\frac{\partial^2 w}{\partial\xi^2}\frac{\partial w}{\partial\eta}+\frac{\partial w}{\partial\xi}\frac{\partial^2 w}{\partial\xi\partial\eta}\right)\right]$$

$$\phi_2(\xi,\eta)=\varepsilon_y-\frac{1}{2}\left(\frac{\partial w}{\partial\eta}\right)^2+\frac{1}{2}(x-\xi)\cdot\left(2\frac{\partial\varepsilon_y}{\partial\xi}-\frac{\partial\gamma_{xy}}{\partial\eta}+\frac{\partial w}{\partial\xi}\frac{\partial^2 w}{\partial\eta^2}-\frac{\partial w}{\partial\eta}\frac{\partial^2 w}{\partial\xi\partial\eta}\right)$$

u_0、v_0、r_0是任意常数．这样，我们已证明了如下定理．

定理 1　若 w, N_x, N_y 及 S_{xy} 满足(2.1)和(2.5)，ε_x、ε_y 及 γ_{xy} 由(2.3)确定，则存在满足(2.2)的单值位移 u 和 v 的充要条件是(2.7)—(2.9)成立．并且当(2.7)—(2.9)成立时，u 和 v 有表达式(2.10)．

2. Ariy 应力函数及其单值性条件

设中面内的体力有势，即存在单值函数 $U(x,y)$，使得

$$f_x=-\frac{\partial U}{\partial x},\quad f_y=-\frac{\partial U}{\partial y}$$

于是，(2.1)成为

$$\frac{\partial(N_x-hU)}{\partial x}+\frac{\partial S_{xy}}{\partial y}=0,\quad \frac{\partial S_{xy}}{\partial x}+\frac{\partial(N_y-hU)}{\partial y}=0 \tag{2.11}$$

当 Ω 是单连区域时，由(2.11)知，存在函数 P 和 Q 使得

$$\frac{\partial P}{\partial x}=-S_{xy},\quad \frac{\partial P}{\partial y}=N_x-hU$$

$$\frac{\partial Q}{\partial x}=N_y-hU,\quad \frac{\partial Q}{\partial y}=-S_{xy}$$

从而存在单值的应力函数 F 使得

$$P=h\frac{\partial F}{\partial y},\quad Q=h\frac{\partial F}{\partial x}$$

亦即

$$N_x=h\frac{\partial^2 F}{\partial y^2}+hU,\quad N_y=h\frac{\partial^2 F}{\partial x^2}+hU,\quad S_{xy}=-h\frac{\partial^2 F}{\partial x\partial y} \tag{2.12}$$

显然，满足(2.12)的应力函数 F 可差一个任意项 $A+Bx+Cy$．当 Ω 是多连通域时，一般说来，这样的单值应力函数是不存在的．下面，我们将给出单值应力函数存在性的一个充分必要条件．

设在 $\bar{\Omega}$ 中存在单值的应力函数 F，并满足(2.12)，则 $Q=h\dfrac{\partial F}{\partial x}$ 和 $P=h\dfrac{\partial F}{\partial y}$ 都是单值的，从而必有

$$\oint_{\Gamma_i}\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy=0,\quad \oint_{\Gamma_i}\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy=0\qquad (i=1,2,\cdots,m)$$

通过一些运算之后，上述等式等价于

$$\oint_{\Gamma_i}\left(N_{ny}+hU\frac{dx}{ds}\right)ds=0 \qquad (i=1,2,\cdots,m) \tag{2.13}$$

$$\oint_{\Gamma_i}\left(N_{nx}-hU\frac{dy}{ds}\right)ds=0 \qquad (i=1,2,\cdots,m) \tag{2.14}$$

其中 s 为 Γ_i 上的弧长参数，$N_{nx}\equiv N_x\cos(n,x)+S_{xy}\cos(n,y)$ $N_{ny}\equiv S_{xy}\cos(n,x)+N_y\cos(n,y)$，$\boldsymbol{n}=(\cos(n,x),\cos(n,y))$ 是 Γ_i 的单位外法线向量．(2.13)、(2.14) 分别表示作用在每一闭围线 Γ_i 上的外力在 x 轴和 y 轴方向上的投影之和为零．当(2.13)、(2.14)成立时，P 和 Q 有如下形式

$$Q(x_1,y_1)=Q(x_0,y_0)-\int_{(x_0,y_0)}^{(x_1,y_1)}\left(N_{ny}+hU\frac{dx}{ds}\right)ds \tag{2.15}$$

$$P(x_1,y_1)=P(x_0,y_0)+\int_{(x_0,y_0)}^{(x_1,y_1)}\left(N_{nx}-hU\frac{dy}{ds}\right)ds \tag{2.16}$$

同时，由 F 的单值性得到

$$\oint_{\Gamma_i}\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=0 \text{ 或者 } \oint_{\Gamma_i}Qdx+Pdy=0, \qquad (i=1,2,\cdots,m)$$

将(2.15)和(2.16)代入上式，分部积分后得到

$$\oint_{\Gamma_i}\left(N_{ny}+hU\frac{dx}{ds}\right)x(s)ds-\oint_{\Gamma_i}\left(N_{nx}-hU\frac{dy}{ds}\right)y(s)ds=0 \tag{2.17}$$

$$(i=1,2,\cdots,m)$$

这表示作用在 Γ_i 上的外力对原点的矩之和为零．反之，不难证明，当 (2.13)、(2.14) 及(2.17)成立时，由表达式

$$\begin{aligned}F(x_1,y_1)=&F(x_0,y_0)+\frac{1}{h}Q(x_0,y_0)(x_1-x_0)+\frac{1}{h}P(x_0,y_0)(y_1-y_0)\\&-\frac{1}{h}\int_{(x_0,y_0)}^{(x_1,y_1)}\left(N_{ny}+hU\frac{dx}{ds}\right)(x_1-x(s))ds\\&+\frac{1}{h}\int_{(x_0,y_0)}^{(x_1,y_1)}\left(N_{nx}-hU\frac{dy}{ds}\right)(y_1-y(s))ds\end{aligned} \tag{2.18}$$

确定的单值函数满足(2.12)．这样，我们证明了下面的定理．

定理 2 如果在 Ω 的每一闭围线 Γ_i 上，外力的合力和合力矩为零，即(2.13)、(2.14)及(2.17)成立，则存在满足(2.12)的单值应力函数 $F(x,y)$．反之亦真．

三、开孔薄板大挠度问题的单值数学模型

1. 控制方程

现在假定 $\partial\Omega$ 上给定的外力满足(2.13)、(2.14) 及(2.17)．于是在 Ω 中就存在满足(2.12)的单值应力函数$F(x,y)$．在(2.1)和(2.5)中消去 N_x，N_y 及 S_{xy} 得到

$$\begin{aligned}G_1(w,F)\equiv &D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}-q\\&-h\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}+U\Delta w\right]=0\end{aligned} \tag{3.1}$$

$$G_2(w,F)\equiv\frac{1}{E_2}\frac{\partial^4F}{\partial x^4}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\frac{\partial^4F}{\partial x^2\partial y^2}+\frac{1}{E_1}\frac{\partial^4F}{\partial y^4}-\left(\frac{\partial^2w}{\partial x\partial y}\right)^2$$

$$+\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}+\frac{1-\nu_2}{E_2}\frac{\partial^2U}{\partial x^2}+\frac{1-\nu_1}{E_1}\frac{\partial^2U}{\partial y^2}=0 \tag{3.1}$$

为了能由(3.1)完全确定 F 和 w，我们还要给出 $\partial\Omega$ 上的边界条件。例如，若 Γ_i 固支，则有

$$w=\frac{\partial w}{\partial n}=0 \tag{3.2}$$

若 Γ_i 简支，则有

$$w=M_n=0 \tag{3.3}$$

若 Γ_i 自由，则有

$$M_n=V_n+\frac{\partial H_{tn}}{\partial s}=0 \tag{3.4}$$

其中

$$\left.\begin{aligned}&M_n=M_x\cos^2(n,x)+M_y\cos^2(n,y)+2H_{xy}\cos(n,x)\cos(n,y)\\&V_n=-\frac{\partial}{\partial x}\left(D_1\frac{\partial^2w}{\partial x^2}+D_3\frac{\partial^2w}{\partial y^2}\right)\cos(n,x)-\frac{\partial}{\partial y}\left(D_3\frac{\partial^2w}{\partial x^2}+D_2\frac{\partial^2w}{\partial y^2}\right)\cos(n,y)\\&H_{tn}=(M_y-M_x)\cos(n,x)\cos(n,y)+H_{xy}(\cos^2(n,x)-\cos^2(n,y))\end{aligned}\right\} \tag{3.5}$$

同时

$$M_x=-D_1\left(\frac{\partial^2w}{\partial x^2}+\nu_2\frac{\partial^2w}{\partial y^2}\right),\quad M_y=-D_2\left(\frac{\partial^2w}{\partial y^2}+\nu_1\frac{\partial^2w}{\partial x^2}\right),$$

$$H_{xy}=-2D_k\frac{\partial^2w}{\partial x\partial y} \tag{3.6}$$

关于板平面内的边界条件则应作如下考虑：设在 $\partial\Omega$ 上给定板平面内的外力，并设它们满足(2.13)、(2.14)及(2.17)。在 Γ_i 上任取一定点 $M_i(i=0,1,2,\cdots,m)$，并取 $M_0(x_0,y_0)$ 为(2.15)、(2.16)及(2.18)中的指定点。不失一般性，令 $Q(x_0,y_0)=P(x_0,y_0)=F(x_0,y_0)=0$，并记

$$\varphi_i(\xi,\eta)=-\frac{1}{h}\int_{(x_i,y_i)}^{(\xi,\eta)}\left(N_{ny}+hU\frac{dx}{ds}\right)(\xi-x)ds+\frac{1}{h}\int_{(x_i,y_i)}^{(\xi,\eta)}\left(N_{nx}-hU\frac{dy}{ds}\right)(\eta-y)ds$$

$$\phi_i(\xi,\eta)=-\frac{1}{h}\int_{(x_i,y_i)}^{(\xi,\eta)}\left(N_{ny}+hU\frac{dx}{ds}\right)ds\cos(n,x)+\frac{1}{h}\int_{(x_i,y_i)}^{(\xi,\eta)}\left(N_{nx}-hU\frac{dy}{ds}\right)ds\cos(n,y) \tag{3.7}$$

$$(i=0,1,2,\cdots,m)$$

其中 $(\xi,\eta)\in\Gamma_i$，积分是沿 Γ_i 取的。由于外力给定，所以 φ_i 和 ϕ_i 是 Γ_i 上的点的已知函数。由(2.15)、(2.16)及(2.18)知，在 Γ_0 上有

$$F=\varphi_0(\xi,\eta),\ \frac{\partial F}{\partial n}=\phi_0(\xi,\eta) \tag{3.8}$$

在 Γ_i 上有

$$F = F(x_i, y_i) + \varphi_i(\xi, \eta) \qquad (i = 1, 2, \cdots, m)$$

$$\frac{\partial F}{\partial n} = \frac{1}{h} Q(x_i, y_i) \cos(n, x) + \frac{1}{h} P(x_i, y_i) \cos(n, y) + \psi_i(\xi, \eta)$$

亦即

$$\begin{aligned} F &= \varphi_i(\xi, \eta) + a_i \\ \frac{\partial F}{\partial n} &= \psi_i(\xi, \eta) + a_{i+m} \cos(n, x) + a_{i+2m} \cos(n, y) \end{aligned} \tag{3.9}$$

其中， $a_i = F(x_i, y_i)$， $a_{i+m} = \frac{1}{h} Q(x_i, y_i), a_{i+2m} = \frac{1}{h} P(x_i, y_i)$

是待定常数.

最后，利用(2.12)，由(2.3)得到

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_1}\left[\frac{\partial^2 F}{\partial y^2} - \nu_1 \frac{\partial^2 F}{\partial x^2} + (1 - \nu_1) U\right] \\ \varepsilon_y &= \frac{1}{E_2}\left[\frac{\partial^2 F}{\partial x^2} - \nu_2 \frac{\partial^2 F}{\partial y^2} + (1 - \nu_2) U\right] \\ \gamma_{xy} &= -\frac{1}{G} \frac{\partial^2 F}{\partial x \partial y} \end{aligned}\right\} \tag{3.10}$$

把(3.10)代入单值性条件(2.7)—(2.9)，得到 $3m$ 个包含 w 和 F 的三阶导数的积分等式. 记这些等式为

$$DC(w, F) = 0 \tag{3.11}$$

其中 $DC: C^3(\bar{\Omega}) \times C^3(\bar{\Omega}) \to R^{3m}$.

这样，我们的基本问题(简称问题(E))是在边界条件(3.2)(或(3.3)，或(3.4))、(3.8)、(3.9)及约束条件(3.11)下，确定满足(3.1)的 w, F 和 $3m$ 维向量 $A = (a_1, a_2, \cdots, a_{3m})$.

2. 问题(E)的解法

本段简述问题(E)的计算方法. 首先，若我们把(3.1)在边界条件(3.2)、(3.8)、(3.9)下的解记为 $w(x, y; A), F(x, y; A)$、并把它们代入(3.11)，则得到关于 $A = (a_1, a_2, \cdots, a_{3m})$的 $3m$ 个方程

$$\mathscr{F}(A) = DC(w(x, y; A), F(x, y; A)) = 0 \tag{3.12}$$

设 A^* 是(3.12)的解，则 $w(x, y; A^*)$、$F(x, y; A^*)$就是问题(E)的解. 但是，(3.12)是关于 A 的非线性方程组，因而对它求解一般要利用迭代法(例如牛顿法). 若 $w(x, y; A)$、$F(x, y; A)$的解析表达式无法精确求得，则 $\mathscr{F}$ 的导数不能简单地计算出来，因而使用迭代法去解(3.12)就发生一定的困难. 这时，与其去推导计算 $\mathscr{F}$ 的导数公式，不如直接对问题(E)采用抽象空间的牛顿迭代程序. 设问题(E)的第 k 次迭代 $w^{(k)}(x, y)$, $F^{(k)}(x, y)$, $A^{(k)}$ 已知，则第 $k+1$ 次迭代 $w^{(k+1)}$、$F^{(k+1)}$、$A^{(k+1)}$ 应由如下线性方程确定:

$$\begin{aligned} &\bar{L}_1 w^{(k+1)} - h([w^{(k+1)}, F^{(k+1)}] + U \Delta w^{(k+1)}) - h[w^{(k)}, F^{(k+1)}] \\ &\quad = q - h[w^{(k)}, F^{(k)}] \end{aligned} \tag{3.13}$$

$$\bar{L}_2 F^{(k+1)} + [w^{(k)}, F^{(k+1)}] = \frac{1}{2}[w^{(k)}, w^{(k)}] - \frac{1 - \nu_2}{E_2} \frac{\partial^2 U}{\partial x^2} - \frac{1 - \nu_1}{E_1} \frac{\partial^2 U}{\partial y^2}$$

其中

$$\left.\begin{aligned}\bar{L}_1 w &\equiv D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}\\ \bar{L}_2 F &\equiv \frac{1}{E_2}\frac{\partial^4 F}{\partial x^4}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{1}{E_1}\frac{\partial^4 F}{\partial y^4}\\ [F,w] &\equiv \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}\end{aligned}\right\}\tag{3.14}$$

边界条件为

$$w^{(k+1)}(\xi,\eta)=\frac{\partial w^{(k+1)}(\xi,\eta)}{\partial n}=0\qquad \text{当 } \partial\Omega \text{ 固定时，} (\xi,\eta)\in\partial\Omega\tag{3.15}$$

$$\left.\begin{aligned}&F^{(k+1)}(\xi,\eta)=a_i^{(k+1)}+\varphi_i(\xi,\eta)\\ &\frac{\partial F^{(k+1)}(\xi,\eta)}{\partial n}=a_{i+m}^{(k+1)}\cos(n,x)+a_{i+2m}^{(k+1)}\cos(n,y)+\psi_i(\xi,\eta)\\ &\qquad (\xi,\eta)\in\Gamma_i(i=1,2,\cdots,m)\end{aligned}\right\}\tag{3.16}$$

$$\left.\begin{aligned}&F^{(k+1)}(\xi,\eta)=\varphi_0(\xi,\eta)\\ &\frac{\partial F^{(k+1)}(\xi,\eta)}{\partial n}=\psi_0(\xi,\eta)\end{aligned}\right\}\quad(\xi,\eta)\in\Gamma_0\tag{3.17}$$

约束条件为

$$\begin{aligned}&DC_w(w^{(k)},F^{(k)})(w^{(k+1)}-w^{(k)})+DC_F(w^{(k)},F^{(k)})(F^{(k+1)}-F^{(k)})\\ &\quad=-DC(w^{(k)},F^{(k)})\end{aligned}\tag{3.18}$$

首先在边界条件 (3.15)—(3.17) 中令 $A^{(k+1)}=0$，并记这时相应方程 (3.13) 的解为 $w_p^{(k+1)}(x,y)$、$F_p^{(k+1)}(x,y)$. 其次在边界条件(3.15)—(3.17)中令

$$A^{(k+1)}=e_j=(\overbrace{0,\cdots,0,1}^{j},0,\cdots,0),$$

并记这时与 (3.13) 相应的齐次方程的解为 $w_j^{(k+1)}(x,y)$, $F_j^{(k+1)}(x,y)$, 则我们得到 (3.13)—(3.17)的解为

$$\left.\begin{aligned}W^{(k+1)}(x,y)&=w_p^{(k+1)}(x,y)+\sum_{j=1}^{3m}a_j^{(k+1)}w_j^{(k+1)}(x,y)\\ F^{(k+1)}(x,y)&=F_p^{(k+1)}(x,y)+\sum_{j=1}^{3m}a_j^{(k+1)}F_j^{(k+1)}(x,y)\end{aligned}\right\}\tag{3.19}$$

将它们代入(3.18)，得到一组关于 $A^{(k+1)}$ 的线性方程组

$$\begin{aligned}&\sum_{j=1}^{3m}(DC_w(w^{(k)},F^{(k)})w_j^{(k+1)}+DC_F(w^{(k)},F^{(k)})F_j^{(k+1)})a_j^{(k+1)}\\ &\quad=-DC(w^{(k)},F^{(k)})+DC_w(w^{(k)},F^{(k)})(w^{(k)}-w_p^{(k+1)})\\ &\qquad+DC_w(w^{(k)},F^{(k)})(F^{(k)}-F_p^{(k+1)})\end{aligned}\tag{3.20}$$

由(3.20)解出 $A^{(k+1)}$，则由(3.19)就可确定第 $k+1$ 次迭代 $w^{(k+1)}$、$F^{(k+1)}$ 和 $A^{(k+1)}$.

关于上述迭代过程的收敛性问题以及数值计算的细节和对一些问题的数值结果将另文报告.

四、开孔薄板的临界载荷和屈曲状态

到目前为止,已有不少文章讨论了薄板临界载荷和屈曲解的计算[5,6],但就本质上讲,这些文章都是讨论的未开孔板问题. 这里虽然可对开 m 个孔的正交各向异性板的临界载荷的算法作出一般的讨论,但为简单起见,我们只对 $m=1$ 的各向同性板进行分析,并设 $U\equiv 0, q\equiv 0$. 这时(2.19)成为通常 Von Kármán 方程

$$\Delta^2 w=\frac{h}{D}[w,F],\quad \Delta^2 F=-\frac{E}{2}[w,w] \tag{4.1}$$

其中 $D=Eh^3/12(1-\nu^2)$. 位移单值性条件(2.7)—(2.9)可写成

$$L_i(F)-N_i(w)=0 \qquad (i=1,2,3) \tag{4.2}$$

式中 $L_i(w)$ 是 F 的线性泛函, $N_i(w)$ 是 w 的齐二次非线性泛函,它们为

$$\begin{cases} L_1(F)\equiv \dfrac{2}{E}\oint_{\Gamma_0}-\dfrac{\partial}{\partial y}\Delta F dx+\dfrac{\partial}{\partial x}\Delta F dy \\ L_2(F)\equiv \dfrac{2}{E}\oint_{\Gamma_0}\dfrac{\partial^2 F}{\partial y^2}dx-\dfrac{\partial^2 F}{\partial x\partial y}dy+\dfrac{2}{E}\oint_{\Gamma_0}-y\dfrac{\partial}{\partial y}\Delta F dx+y\dfrac{\partial}{\partial x}\Delta F dy \\ L_3(F)\equiv \dfrac{2}{E}\oint_{\Gamma_0}-\dfrac{\partial^2 F}{\partial x\partial y}dx+\dfrac{\partial^2 F}{\partial x^2}dy+\dfrac{2}{E}\oint_{\Gamma_0}x\dfrac{\partial}{\partial y}\Delta F dx-x\dfrac{\partial}{\partial x}\Delta F dy \end{cases}$$

$$\begin{cases} N_1(w)\equiv \oint_{\Gamma_0}\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}\right)dx \\ \qquad -\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}-\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}\right)dy \\ N_2(w)\equiv \oint_{\Gamma_0}y\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}\right)dx-y\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}-\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}\right)dy \\ \qquad +\oint_{\Gamma_0}\left(\dfrac{\partial w}{\partial x}\right)^2 dx+\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial x}dy \\ N_3(w)\equiv -\oint_{\Gamma_0}x\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}\right)dx \\ \qquad -x\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}-\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}\right)dy+\oint_{\Gamma_0}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}dx+\left(\dfrac{\partial w}{\partial y}\right)^2 dy \end{cases}$$

关于 w 的边界条件仍保持不变,例如对固支边为

$$w=\frac{\partial w}{\partial n}=0 \qquad 在\ \partial\Omega\ 上 \tag{4.3}$$

现在,设在 $\partial\Omega$ 上给定板平面内的外力,并设它们和某一标量 λ 成比例,且满足(2.13)、(2.14)及(2.17). 于是,(3.8)、(3.9)成为

$$F=\lambda\varphi_0,\quad \frac{\partial F}{\partial n}=\lambda\phi_0,\quad 在\ \Gamma_0\ 上 \tag{4.4}$$

$$F=\lambda\varphi_1+a_1,\quad \frac{\partial F}{\partial n}=\lambda\phi_1+a_2\cos(n,x)+a_3\cos(n,y),\ 在\ \Gamma_1\ 上 \tag{4.5}$$

若板在面内边界力(4.4)、(4.5)的作用下只发生面内的变形,即 $w\equiv 0$,则称板处于未屈曲状态. 对于未屈曲状态的解的计算,可由在(4.1)—(4.5)中令 $w\equiv 0$ 来得到. 于是要求

的 F 应满足

$$\Delta^2 F = 0 \tag{4.6}$$

$$L_i(F) = 0, \quad (i = 1, 2, 3) \tag{4.7}$$

(4.4)—(4.7) 就是通常多连通域的平面应力问题．利用复函数方法可以计算出它的解．令 $\lambda = 1$ 时，(4.4)—(4.7)的解为 $F^*, A^* = (a_1^*, a_2^*, a_3^*)$，则显然对任意的 λ，(4.1)—(4.5)有未屈曲状态解

$$w \equiv 0, \ F = \lambda F^*, \ A = \lambda A^* \tag{4.8}$$

引进变换

$$\hat{F} = F - \lambda F^*, \ \hat{A} = A - \lambda A^* \tag{4.9}$$

则(4.1)—(4.5)成为

$$\left.\begin{aligned} \Delta^2 w &= \frac{h}{D}[w, \hat{F}] + \lambda \frac{h}{D}[w, F^*] \\ \Delta^2 F &= -\frac{E}{2}[w, w] \end{aligned}\right\} \tag{4.10}$$

$$w = \frac{\partial w}{\partial n} = 0, \text{在 } \partial\Omega \text{ 上} \tag{4.11}$$

$$\left.\begin{aligned} &\hat{F} = \frac{\partial \hat{F}}{\partial n} = 0, \text{ 在 } \Gamma_0 \text{ 上} \\ &\hat{F} = \hat{a}_1, \ \frac{\partial \hat{F}}{\partial n} = \hat{a}_2 \cos(n, x) + \hat{a}_3 \cos(n, y) \end{aligned}\right\} \tag{4.12}$$

$$L_i(\hat{F}) - N_i(w) = 0 \quad (i = 1, 2, 3) \tag{4.13}$$

对任意λ，(4.10)—(4.13)显然有平凡解 $w \equiv 0, \hat{F} \equiv 0$ 及 $\hat{A} = 0$，这相应于 (4.1)—(4.5)的未屈曲状态．按一般非线性理论知，使得 (4.1)—(4.5) 的未屈曲状态发生屈曲分叉的 λ (即临界载荷)必定使 (4.10)—(4.13) 在平凡解处的线性化问题有非零解，即 λ 应使下述问题有非零解：

$$\Delta^2 w = \frac{h}{D}\lambda[w, F^*], \ \Delta^2 \hat{F} = 0 \tag{4.14$_{a,b}$}$$

$$L_i(\hat{F}) = 0 \tag{4.15}$$

以及边界条件(4.11)和(4.12)．由$(4.14)_b$，(4.12) 及(4.15)得 $\hat{F} \equiv 0, \hat{A} = 0$．因此，临界载荷就是问题 $(4.14)_a$ 及 (4.11) 的特征值． 注意到 $(4.14)_a$、(4.11) 和位移单值性条件(4.13)无关．因此我们看到，对临界载荷来讲，开孔板和不开孔板的数学模型是相同的．因而，对不开孔板的特征值问题的数学理论和计算方法[5,6] 对开孔板也是适用的．虽然开孔会引起临界载荷和特征函数在数值上的较大变化．

下面我们来讨论在临界载荷 $\bar{\lambda}$ 附近非线性问题(4.10)—(4.13)的非零解．设 $\bar{\lambda}$ 是$(4.14)_a$、(4.11)的单重特征值[7]，$\bar{w}$ 是相应的特征函数，且

$\iint\limits_{\Omega} \bar{w}^2 da = 1$，则按一般分支理论知，(4.10)—(4.13)的平凡解在 $\bar{\lambda}$ 附近必发生分叉．为了计算这个分叉解支，我们作变换：

$$\left.\begin{aligned} w &= \varepsilon\bar{w}(x,y) + \varepsilon w_1(x,y) \\ \hat{F} &= \varepsilon F_1(x,y) \\ A &= \varepsilon\bar{A}, \ \lambda = \bar{\lambda} + \lambda_1 \end{aligned}\right\} \tag{4.16}$$

其中 ε 为小参数，为

$$\varepsilon = \iint_{\Omega} \bar{w} w da \tag{4.17}$$

因而

$$\iint_{\Omega} w_1 \bar{w} da = 0 \tag{4.18}$$

将(4.16)代入(4.10)—(4.13)得到

$$\left.\begin{aligned} \Delta^2 w_1 &= \varepsilon \frac{h}{D}[\bar{w} + w_1, F_1] + \frac{h}{D}\lambda_1[w_1 + \bar{w}, F^*] + \frac{h}{D}\bar{\lambda}[w_1, F^*] \\ \Delta^2 F_1 &= -\frac{E}{2}\varepsilon[\bar{w} + w_1, \bar{w} + w_1] \end{aligned}\right\} \tag{4.19}$$

$$w_1 = \frac{\partial w_1}{\partial n} = 0, \text{在 } \partial\Omega \text{ 上} \tag{4.20}$$

$$\left.\begin{aligned} &F_1 = \frac{\partial F_1}{\partial n} = 0 \\ &F_1 = \bar{a}_1, \ \frac{\partial F_1}{\partial n} = \bar{a}_2\cos(n, x) + \bar{a}_3\cos(n, y) \end{aligned}\right\} \text{在 } \Gamma_0 \text{ 上} \tag{4.21}$$

$$L_i(F_1) - \varepsilon N_i(\bar{w} + w_1) = 0 \qquad (i = 1, 2, 3) \tag{4.22}$$

在问题(4.18)—(4.22)中，基本未知量是 $w_1, F_1, \bar{A}$ 和 λ_1，而 ε 是参数．显然，当 $\varepsilon = 0$ 时，(4.19)—(4.23)有一个零解 $w_1 \equiv 0$，$F_1 \equiv 0$，$\bar{A} = 0$，$\lambda_1 = 0$．这个解处的线性化问题为

$$\Delta^2 w_1 = \frac{h}{D}\lambda_1[\bar{w}, F^*] + \frac{h}{D}\lambda[w_1, F^*] \tag{4.23}$$

$$\Delta^2 F_1 = 0 \tag{4.24}$$

$$L_i(F_1) = 0 \qquad (i = 1,2,3) \tag{4.25}$$

以及(4.18)、(4.20)、(4.21)．由于 $\bar{\lambda}$ 是$(4.14)_a$、(4.11)的单特征值，因而上述问题只有零解．由椭圆型算子的一般理论[7]知，(4.18)—(4.22)在 $\varepsilon = 0$ 附近存在唯一解：

$$w_1 = w_1(x,y,\varepsilon), \ F_1 = F_1(x,y,\varepsilon), \ \bar{A} = \bar{A}(\varepsilon), \ \lambda_1 = \lambda_1(\varepsilon) \tag{4.26}$$

且满足

$$w_1(x,y,0) = F_1(x,y,0) = 0, \ \bar{A}_1(0) = 0, \quad \lambda_1(0) = 0$$

并且对小的 ε，可通过牛顿迭代过程来求解(4.26)．这样，虽然我们得到了与不开孔板[5,6]的类似结论，但值得注意的是，由于在开孔板的公式中出现了单值性条件(4.22)，因而用牛顿法去求解(4.18)—(4.22)时，要比不开孔板的情形更困难．开孔对重特征值及二次分叉的影响将另外讨论．

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GENERAL MATHEMATICAL THEORY OF LARGE DEFLECTIONS OF THIN PLATES WITH SOME HOLES

Zhu Zhengyou Cheng Changjun
(Lanzhou University)

Abstract

It is well known that the theoretical analysis and the practical computations for the large deflection of the thin plate with some holes become more difficult due to the multivalued property of the stress function F and the singlevalued demands on the displacement. The necessary and sufficient conditions which assure that F is single-valued are obtained in this paper. Then we prove that the single-valued demands on the displacement is equivalent to some constrained equations $Li(F) - Ni(w) = 0$, $i = 1, \ldots, 3m$, where $Li(F)$ is the linear functional in F and $Ni(w)$ is the quadratic homogeneous functional in w. From these conclusions, the single-valued governing equations for the plate with some holes are derived. It is a fourth order partial differential system with 3 m indeterminate constants and 3 m constrained equations. We present a numerical method for solving this problem.

The problem of the critical load is considered and if the critical load λ is 'single', the iterative scheme for computing the buckled states that branch from trivial solution is given.