

最小二乘配点法解薄板几何非线性弯曲问题

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用最小二乘配点法分析薄板弯曲问题,国内于1978年首先由徐次达、施德芳用幂级数作为挠度试函数进行了解算^[1]。1979年何广乾、张维岳成功地以最小二乘边界配点法解算了壳体的线性弯曲问题^[2]。本文系用最小二乘内部配点法解算薄板几何非线性弯曲问题,所用挠度试函数为双三角级数,以边界可动简支方形柔韧板为例进行计算,在解非线性方程组时采用 Levenberg-Margardt 法(阻尼最小二乘法)^[4],克服方程组的奇异性和病态。计算结果与 Levy 的成果^[3]进行了比较。

1. 定解微分方程和试函数:

薄板几何非线性弯曲问题的定解微分方程是:

$$D\nabla^4 w - tL(w, \Phi) = q \quad (1.1)$$

$$\frac{1}{E} \nabla^4 \Phi + \frac{1}{2} L(w, w) = 0 \quad (1.2)$$

式中

$$D = \frac{Et^3}{12(1-\mu^2)} \quad (1.3)$$

为抗弯刚度, E 为弹性模量, μ 为泊松比, t 为板厚, q 是横向分布力集度, w 是中面挠度函数, Φ 为应力函数,也就是中面力与 Φ 有如下关系:

$$N_x = t \cdot \frac{\partial^2 \Phi}{\partial y^2}, \quad N_y = t \cdot \frac{\partial^2 \Phi}{\partial x^2}, \quad N_{xy} = -t \cdot \frac{\partial^2 \Phi}{\partial x \partial y} \quad (1.4)$$

式(1.1)与(1.2)中的非线性项是:

$$L(w, \Phi) = \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial^2 \Phi}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 \Phi}{\partial x \partial y} \quad (1.5)$$

$$\frac{1}{2} L(w, w) = \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \quad (1.6)$$

对于四边简支而边界可以自由接近的方板,我们选择试函数(即挠度和应力函数的近似表达式)为:

$$\tilde{w}(x, y) = \sum_m \sum_n A_{mn} \sin m\pi x \sin n\pi y \quad (1.7)$$

$$\tilde{\Phi}(x, y) = \sum_s \sum_t B_{st} \sin s\pi x \sin t\pi y \quad (1.8)$$

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这样,在 $x = \text{const}$ 边上的

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad N_x = 0 \quad (1.9)$$

得到了满足,同样 $y = \text{const}$ 边上的相应条件亦已满足.

2. 计算方法

首先将试函数 (1.7)、(1.8) 代入卡门方程 (1.1)、(1.2) 形成残数方程式,接着在内部配 K 个点,即得到 $2K$ 个残数方程,可用最小二乘法求待定系数 A_{mn} 及 B_{ss} 的值.

为了使残数平方和 $R^T R$ (R 为残数列矩阵) 为极小值,可用求非线性最小二乘问题最优的方法. 本文采用了 Levenberg Marguardt 法(亦称为阻尼最小二乘法)^[4], 现简述于下: 设目标函数为

$$S(\mathbf{C}) = \mathbf{R}^T(\mathbf{C})\mathbf{R}(\mathbf{C}) \quad (2.1)$$

其中

$$\mathbf{C}^T = [c_1, c_2, c_3, \dots, c_n] \quad (2.2)$$

为解向量,

$$\mathbf{R}(\mathbf{C})^T = [R_1(\mathbf{C}), R_2(\mathbf{C}), \dots, R_m(\mathbf{C})] \quad (2.3)$$

为向量值函数. 为了求目标函数 $S(\mathbf{C})$ 的极小值,必须使

$$\nabla S(\mathbf{C}) = \frac{\partial S(\mathbf{C})}{\partial \mathbf{C}} = 0 \quad (2.4)$$

即

$$\begin{aligned} \nabla S(\mathbf{C}) &= R_1(\mathbf{C})\nabla R_1(\mathbf{C}) + \dots + R_m(\mathbf{C})\nabla R_m(\mathbf{C}) \\ &= \begin{bmatrix} \frac{\partial R_1}{\partial c_1} & \frac{\partial R_1}{\partial c_2} & \dots & \frac{\partial R_1}{\partial c_n} \\ \frac{\partial R_2}{\partial c_1} & \frac{\partial R_2}{\partial c_2} & \dots & \frac{\partial R_2}{\partial c_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial R_m}{\partial c_1} & \frac{\partial R_m}{\partial c_2} & \dots & \frac{\partial R_m}{\partial c_n} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{Bmatrix} = 0 \end{aligned} \quad (2.5)$$

将上式简写为:

$$\nabla S(\mathbf{C}) = \mathbf{DR}(\mathbf{C})^T \mathbf{R}(\mathbf{C}) = 0 \quad (2.6)$$

式中

$$\mathbf{DR}(\mathbf{C})^T = \begin{bmatrix} \frac{\partial R_1}{\partial c_1} & \frac{\partial R_1}{\partial c_2} & \dots & \frac{\partial R_1}{\partial c_n} \\ \frac{\partial R_2}{\partial c_1} & \frac{\partial R_2}{\partial c_2} & \dots & \frac{\partial R_2}{\partial c_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial R_m}{\partial c_1} & \frac{\partial R_m}{\partial c_2} & \dots & \frac{\partial R_m}{\partial c_n} \end{bmatrix} \quad (2.7)$$

将 $\mathbf{R}(\mathbf{C})$ 在 \mathbf{C}_k 附近作一级泰勒展开如下:

$$\mathbf{R}(\mathbf{C}) = \mathbf{R}(\mathbf{C}_k) + \mathbf{A}_k \Delta \mathbf{C}_k \quad (2.8)$$

其中

$$\mathbf{A}_k = \mathbf{DR}(\mathbf{C}_k) \quad (2.9)$$

将(2.8)代入式(2.6)可以求得下式:

$$\mathbf{A}_k^T \mathbf{A}_k \Delta \mathbf{C}_k = -\mathbf{A}_k^T \mathbf{R}(\mathbf{C}_k) \quad (2.10)$$

为了保证 $\mathbf{A}_k^T \mathbf{A}_k$ 非奇异, Levenberg-Marguardt 法通过增大矩阵主对角线元素, 将(2.10)式改造成下式:

$$(\mathbf{A}_k^T \mathbf{A}_k + p\mathbf{I}) \Delta \mathbf{C}_k = -\mathbf{A}_k^T \mathbf{R}(\mathbf{C}_k) \quad (2.11)$$

其中 p 为正实参数, 称为阻尼因子, \mathbf{I} 为单位矩阵. 这样就减弱了方程(2.10)的病态性. 于是得到如下的迭代公式:

$$\left. \begin{aligned} \mathbf{C}_{k+1} &= \mathbf{C}_k + \Delta \mathbf{C}_k \\ \Delta \mathbf{C}_k &= -(\mathbf{A}_k^T \mathbf{A}_k + p\mathbf{I})^{-1} \mathbf{A}_k^T \mathbf{R}(\mathbf{C}_k) \end{aligned} \right\} \quad (2.12)$$

($k = 0, 1, 2, \dots$)

3. 数值结果

将试函数中的待定系数 A_{mn} 及 B_{st} 合并成一个解向量:

$$\begin{aligned} \mathbf{C}^T &= [c_1, c_2, \dots, c_{m \times n + s \times t}] \\ &= [A_{11}, A_{12}, \dots, A_{mn}, B_{11}, B_{12}, \dots, B_{st}] \end{aligned} \quad (3.1)$$

用上节所述方法求解, \mathbf{C} 的迭代初值 \mathbf{C}_0 由 A_{mn} 及 B_{st} 作为相应的线性解时的值充任. 在计算时试函数 \tilde{w} 及 $\tilde{\Phi}$ 所取项数等于板中配点数. 配点方法是用均匀配点法^[1].

本文用几种不同的配点数, 在各级载荷情况下进行了计算, 较大的荷载可使板中心挠度达到厚度的 3 倍以上. 计算所得结果示于图 1、图 2、图 3 中. 图 1 表示板的中心挠度

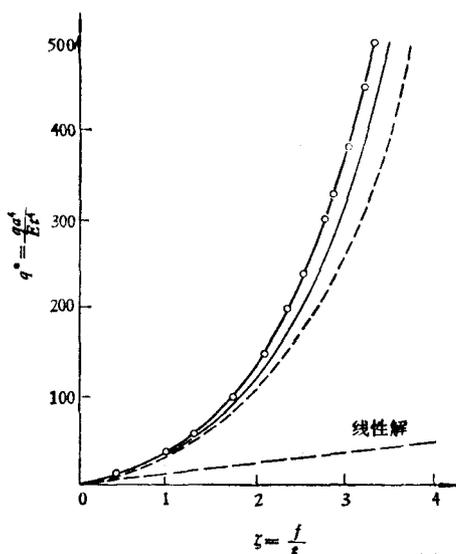


图 1 中点挠度与荷载关系图

—○— 本方法 5×5 配点解 ——Levy 较精确解
----Levy 一级近似解

与所加荷载的关系, 与 Levy 近似解^[4]所得曲线比较, 差别仅 5% 左右. 图 2 表示板中点处中面正应力的值, 与 Levy 近似解相当吻合.

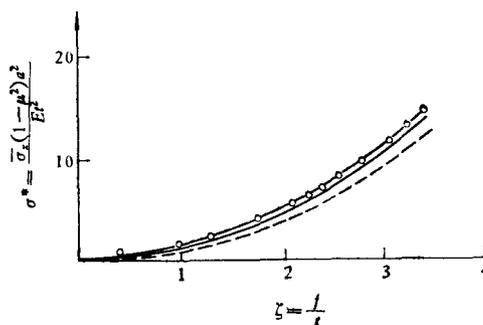


图 2 中曲面应力图

—○— 本方法 25 点配点解 ——Levy 较精确解
----Levy 一级近似解

图 3 示本方法计算所得之中点弯曲应力值与 Levy 解的相应值进行比较. 从图中可见, 本方法的解当试函数取 25 项(配点数为 25) 时与 Levy 解有一定的差别, 但若只取 4

项,却与 Levy 解符合得较好,一般误差小于 1%。估计这是由于 Levy 解得出时所取项数较少,而三角级数的二阶导数又收敛太慢,取少数几项精度不够所致。

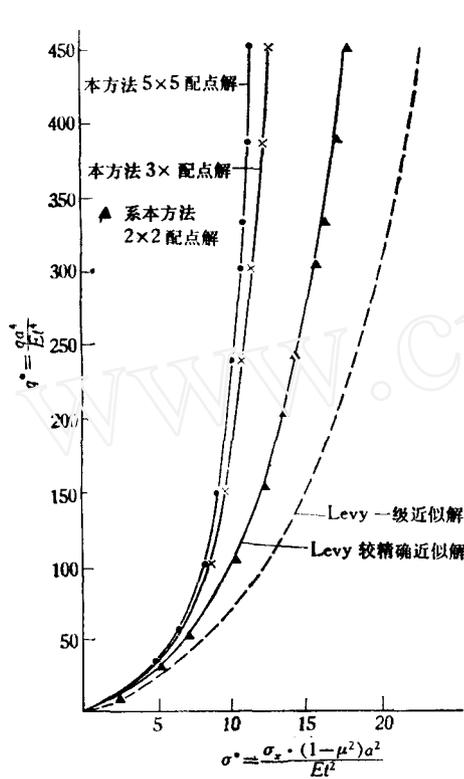


图 3 弯曲应力图

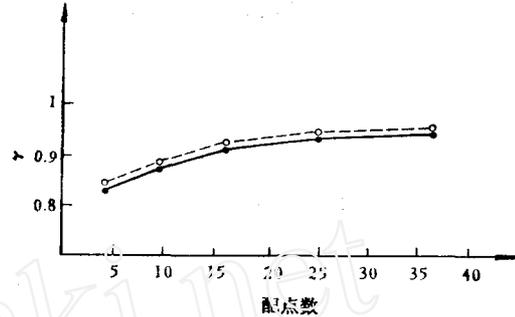


图 4 挠度随配点数(即试函数项数)变化情况
—●— $q^* = 100$, —○— $q^* = 300$

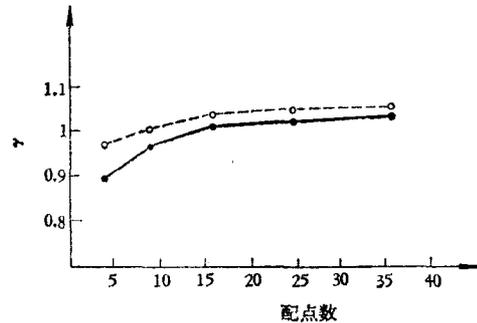


图 5 中面应力随配点数(即试函数项数)变化情况
—○— $q^* = 300$, —●— $q^* = 100$

为了考察本问题解的稳定性,笔者用 4、9、16、25、36 等各种不同的配点数,即试函数分别取 4 项、9 项、16 项、25 项、36 项等不同的项数对各级荷载进行了计算,发现解稳定地趋于某一定值,随着试函数项数增加,解的变化越来越小。我们考虑了 $q^* = 100$ 及 $q^* = 300$ 两种情况(图 4、图 5)。图中配点数与所取试函数的项数是一致的, γ 表示本方法之解与 Levy 解之比。由于 Levy 解也是近似解,所以 γ 不收敛于 1,这是可以理解的,但据此我们已可看出收敛的趋势了。

4. 结论

利用试函数,用最小二乘配点法解板的几何非线性问题在国内尚属初次尝试,本方法具有便捷、计算机程序易编,方法与原理统一等优点^[6]。本方法与有限元法相比,输入工作量大为减少,只要输入几个数据即可,十分方便。解题时间也较省。可克服有限元法要求机器贮量大、计算时间长等缺点。

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ANALYSIS OF GEOMETRICALLY NONLINEAR PLATE-BENDING PROBLEMS BY THE LEAST SQUARES COLLOCATION METHOD

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Abstract

In this paper, the method of discretized least squares collocation has been used successfully by the authors in geometrically nonlinear plate-bending problems. The trial function used is the beam function which satisfies the boundary conditions of rectangular plates. Thus it is only necessary to dispose the residuals in the interior of the plate, the whole problem being in the sense of interior method of M. W. R.

In order to minimize the sum of squares of the residuals, a series of nonlinear algebraic equations have to be solved by the Levenberg-Marquardt Method, which avoids the ill-conditioning of the equations and the iteration process is carried out successfully.

As an example, a square plate hinged at four edges with uniformly located load is studied in this paper. It is found that the precision of calculation of displacement and force factors are satisfactory. Computational results in this case, as compared with that from Levy's solutions, show a difference of less than 5%.

The use of the discretized least squares method in solving geometrically nonlinear plate-bending problems is perhaps new. This method has the advantages of simpler computer program, less work and less time consumption. And much less preparatory work is needed, as compared with the standard Finite Element Method. Only five or four input data are needed. Thus this method are efficient in reducing computer storage and computing time requirements.