非保守场守恒定理及某类连续介质力学的守恒律

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提要　本文通过微分变分原理导出非保守场守恒定理及给出某类连续介质力学的守恒定律．从某种角度看是对文献 [1] 的推广和补充，同时也对文献 [1] 的某些结果作了商榷．

一、非保守场守恒定理

取场变量 $\boldsymbol{q}$，Lagrange 密度 L，非保守项 $\boldsymbol{Q}$．用矩阵方程，变量以一列阵或行阵表示，下角出现张量附标则按重号求和规则进行．由精确到一阶项的无限小变换算子 Δ、变分算子 δ、对多维直线坐标 x_i 全微商算子“$,i$”间的关系 (1) 式，通过无限小量 ε 引进变换函数 (2) 式(包括规范变换函数 $P_i(\boldsymbol{q}, \boldsymbol{q}_{,i}, x_i)$)，转换微分变分原理 (3) 式，从而得定理：

$$\Delta(\ \ ) = \delta(\ \ ) + (\ \ )_{,i}\Delta x_i \tag{1}$$

$$(\text{因而 } [\Delta(\ \ )]_{,i} = \Delta(\ \ )_{,i} + (\ \ )_{,k}(\Delta x_k)_{,i})$$

$$\Delta \boldsymbol{q} = \varepsilon \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{q}_{,i}, x_i),\ \Delta x_i = \varepsilon f_i(\boldsymbol{q}, \boldsymbol{q}_{,i}, x_i) \tag{2}$$

$$\left.\begin{aligned}
&\int\left[\frac{\partial L}{\partial \boldsymbol{q}} - \left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i} + \boldsymbol{Q}\right]\delta \boldsymbol{q} dx = \varepsilon\int(Y - W_{i,i})dx \\
&\quad = \varepsilon\int[Y^* - W_{i,i} + \boldsymbol{Q}(\boldsymbol{F} - \boldsymbol{q}_{,i}f_i)]dx = 0 \\
&Y = \left(\frac{\partial L}{\partial \boldsymbol{q}} + \boldsymbol{Q}\right)\boldsymbol{F} + \frac{\partial L}{\partial \boldsymbol{q}_{,i}}\boldsymbol{F}_{,i} + \left(\frac{\partial L}{\partial x_i} - \boldsymbol{Q}\boldsymbol{q}_{,i}\right)f_i \\
&\qquad + \left(L\delta_{ji} - \frac{\partial L}{\partial \boldsymbol{q}_{,j}}\boldsymbol{q}_{,i}\right)f_{i,j} - P_{i,i} \\
&Y^* = Y - \boldsymbol{Q}(\boldsymbol{F} - \boldsymbol{q}_{,i}f_i) \\
&W_i = \left(L\delta_{ij} - \frac{\partial L}{\partial \boldsymbol{q}_{,i}}\boldsymbol{q}_{,j}\right)f_j + \frac{\partial L}{\partial \boldsymbol{q}_{,i}}\boldsymbol{F} - P_i
\end{aligned}\right\} \tag{3}$$

广义 Noether 定理 1：下列依次为微分形式的场方程、变换律、守恒律的 (4.1)—(4.3) 三式中，满足了其中二式，则第三式必成立；对应积分形式，则是微分变分原理、变换律与守恒律的作用积分的 (4.1)′—(4.3)′ 三式中，成立其中之二，则第三必成立；对应积分形式，则是微分变分原理、变换律与守恒律的作用积分的 (4.1)′—(4.3)′ 三式中，成立其中

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之二，则第三必成立.

$$\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}+\boldsymbol{Q}=0 \tag{4.1}$$

$$Y=0 \tag{4.2}$$

$$W_{i,i}=0 \tag{4.3}$$

$$\int\left[\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}+\boldsymbol{Q}\right]\delta q\, dx=0 \tag{4.1$'$}$$

$$\int Y dx=0 \tag{4.2$'$}$$

$$\int W_{i,i} dx=0 \tag{4.3$'$}$$

广义 Noether 定理 2：下列依次为微分形式的场方程、变换律、守恒律的 (5.1)—(5.3) 三式中，满足了其中二式，则第三式必成立；对应积分形式，则是微分变分原理、变换律与守恒律的作用积分的 (5.1)′—(5.3)′ 三式中，成立其中之二，则第三必成立.

$$\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}+\boldsymbol{Q}=0 \tag{5.1}$$

$$Y^*=0 \tag{5.2}$$

$$W_{i,i}-\boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)=0 \tag{5.3}$$

$$\int\left[\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}+\boldsymbol{Q}\right]\delta \boldsymbol{q} dx=0 \tag{5.1$'$}$$

$$\int Y^* dx=0 \tag{5.2$'$}$$

$$\int[W_{i,i}-\boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)]dx=0 \tag{5.3$'$}$$

这二条定理由 (3) 式即可证明. (4.3) 式是散度形式，(5.3) 式可称谓拟散度形式. (4.2) 式改写为

$$\left.\begin{aligned}&\left(\frac{\partial L}{\partial \boldsymbol{q}}+\boldsymbol{Q}\right)\boldsymbol{F}+\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}}\boldsymbol{q}_{,i}+\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\frac{\partial \boldsymbol{F}}{\partial x_i}+\left(\frac{\partial L}{\partial x_i}-\boldsymbol{Q}\boldsymbol{q}_{,i}\right)f_i\\&\quad+\left(L\frac{\partial f_i}{\partial \boldsymbol{q}}-\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\boldsymbol{q}_{,k}\frac{\partial f_k}{\partial \boldsymbol{q}}\right)\boldsymbol{q}_{,i}+L\frac{\partial f_i}{\partial x_i}-\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\boldsymbol{q}_{,k}\frac{\partial f_k}{\partial x_i}\\&\quad-\frac{\partial P_i}{\partial \boldsymbol{q}}\boldsymbol{q}_{,i}-\frac{\partial P_i}{\partial x_i}=0\\&\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}_{,j}}-\boldsymbol{q}_{,k}\frac{\partial f_k}{\partial \boldsymbol{q}_{,j}}\right)+L\frac{\partial f_i}{\partial \boldsymbol{q}_{,j}}-\frac{\partial P_i}{\partial \boldsymbol{q}_{,j}}=0\end{aligned}\right\} \tag{4.2*}$$

由于 $\boldsymbol{F}, f_i, P_i$ 不依赖于 $\boldsymbol{q}_{,ij}$，比较 (4.2) 式中 $\boldsymbol{q}_{,ij}$ 的同次项系数即可得上式，这里称谓场论的广义 Killing 方程组. 它是关于 $(\boldsymbol{q}, \boldsymbol{q}_{,i}, x_i)$ 空间的变换函数 $\boldsymbol{F}, f_i, P_i$ 所要满足的变换律(或通过场方程 (4.1) 消去 $\boldsymbol{q}_{,ij}$ 由(4.2) 式也可). (4.2)* 式中取 $\boldsymbol{Q}=0$，则把 (5.2) 式改写为 (5.2)* 式. (4.2)′, (5.2)′ 式改写为

$$\begin{aligned}\varepsilon\int Y dx&=\int[\Delta L+L(\Delta x_i)_{,i}+\varepsilon \boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)-\varepsilon P_{i,i}]dx\\&=\Delta\int L dx+\varepsilon\int \boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)dx-\varepsilon\int P_{i,i}dx=0\end{aligned} \tag{4.2$'^*$}$$

$$\varepsilon\int Y^{*}dx=\Delta\int Ldx-\varepsilon\int P_{i,i}dx=0 \tag{5.2)'*}$$

其中用到 $\Delta\int Ldx=\int[\Delta Ldx+L\Delta(dx)],\Delta(dx)=(\Delta x_i)_{,i}dx$. 这二式说明作用量的变换律或对称性. 对保守系: $\boldsymbol{Q}=0$, 这二条定理合一, 并为 Noether 定理; 再取 $P_i=0$, (4.2)'*, (5.2)'* 说明作用量关于无限小变换的不变性.

以上结果是,以场方程、变换律、守恒律三者的等价关系(包括微分形式与积分形式)表达守恒定理;给出散度形式守恒律与拟散度形式守恒律的二种守恒定理;说明微分变分原理是构成守恒定理的依据,可统一导出各种形式及各种情况的守恒定理;得到的非保守场守恒定理是一种普遍性的情况.

保守外力作为非保守项的特殊情况, 定理 2 包括了文献 [1] 的计及保守体力的守恒定理,虽然文献[1]在提法上与这里不同,用这里的符号(不计规范变换函数 P_i), 文献 [1] 建立守恒定理的依据是

$$\Delta\int Ldx-\varepsilon\int \boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)dx=0 \tag{6}$$

并定义为 $\int Ldx$ 的散度不变性,再转化为

$$\int\left[\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}\right]\delta\boldsymbol{q}dx+\varepsilon\int[W_{i,i}-\boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)]dx=0 \tag{7}$$

因此当满足 (8) 式时将成立 (9) 式:

$$\frac{\partial L}{\partial \boldsymbol{q}}-\left(\frac{\partial L}{\partial \boldsymbol{q}_{,i}}\right)_{,i}=0 \tag{8}$$

$$W_{i,i}-\boldsymbol{Q}(\boldsymbol{F}-\boldsymbol{q}_{,i}f_i)=0 \tag{9}$$

守恒律 (9) 式与 (5.3) 式相同. 若 (8) 式改为 (5.1) 式, 则对应地 (6) 式将改为 (5.2)'* 式.

二、某类连续介质力学的守恒律

用于三维空间域及第四维时间 t 域,下标希文表示四维时空域及西文表示三维空域,有时以( )˙表示对时域全微商. 我们把微极线弹性理论的基本方程(例如见文献 [1]) 扩展为普遍形式. 因为有某些一般意义,在扩大了守恒律物理内容的同时,守恒律将更有助于构造某类连续介质力学的场方程. 令 $\boldsymbol{q}=\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{\theta}\end{pmatrix}, \boldsymbol{Q}=(\boldsymbol{Q}_u,\boldsymbol{Q}_\theta), \boldsymbol{F}=\begin{pmatrix}\boldsymbol{F}_u\\ \boldsymbol{F}_\theta\end{pmatrix}$, $\boldsymbol{u}$ 及 $\boldsymbol{\theta}$ 为广义的线位移及角位移, 对应地 $\boldsymbol{e}$ 与 $\boldsymbol{\mu}$ 为广义的线应变与角应变, $\boldsymbol{\sigma}$ 与 $\boldsymbol{m}$ 为广义的应力与偶应力, V 为广义的应变势能, T 为广义的动能. 阵变量的元是三维张量的分量, 称谓"广义量"是指例如 $\boldsymbol{u}$、$\boldsymbol{\theta}$ 由多种三维向量组成. 场方程是

$$\left.\begin{aligned}&L=V(x_\alpha,\ \boldsymbol{e},\ \boldsymbol{\mu})-T(x_\alpha,\ \dot{\boldsymbol{u}},\ \dot{\boldsymbol{\theta}})\\&-\left(\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\right)_{,i}+\left(\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\right)^{\cdot}+\boldsymbol{Q}_u=0\\&\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\theta}}-\left(\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,i}}\right)_{,i}+\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right)^{\cdot}+\boldsymbol{Q}_\theta=0\end{aligned}\right\} \tag{10}$$

其中满足广义物性式：$\boldsymbol{\sigma}=\frac{\partial V}{\partial \boldsymbol{e}}$，$\boldsymbol{m}=\frac{\partial V}{\partial \boldsymbol{\mu}}$；及广义几何式：$\boldsymbol{e}=\boldsymbol{e}(\boldsymbol{\theta},\ \boldsymbol{u}_{,i})$，$\boldsymbol{\mu}=\boldsymbol{\mu}(\boldsymbol{\theta}_{,i})$. 很明显，这些广义量取成最简单的形式，例如可以是微极线弹性理论. 因此这个普遍模型把微极线弹性理论作为特殊情况包括在内. 建立守恒律（5.3）式. 取变换函数

$$f_i=\nu x_i+\varepsilon_{ijk}x_jb_k+c_i \tag{11.1}$$

$$f_4=\nu t+c_4 \tag{11.2}$$

$$\boldsymbol{F}_u=-\boldsymbol{u}\nu+\boldsymbol{u}\varepsilon_kb_k+\boldsymbol{a}\varepsilon_kx_k+\boldsymbol{d} \tag{11.3}$$

$$\boldsymbol{F}_\theta=\boldsymbol{a}-\boldsymbol{\theta}\nu \tag{11.4}$$

$$P_\alpha=0 \tag{11.5}$$

其中 $\boldsymbol{a},b_k,c_k,\boldsymbol{d},\nu$ 为常量. ε_k 是与 ε_{ijk} 相同的交叉张量，例如 $\boldsymbol{u}\,\varepsilon_k\,b_k$ 的阵元是 $u_i\,\varepsilon_{kij}b_k$，写成 ε_k 是为了算式中均衡附标.（11）代入（5.3）式比较系数 $\boldsymbol{d},\boldsymbol{a},c_4,c_i,\nu,b_k$ 依次得守恒律

$$\left(\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\right)_{,i}-\left(\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\right)^{\cdot}-\boldsymbol{Q}_u=0 \tag{12.1}$$

$$\boldsymbol{\varepsilon}_i\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}+\left(\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,i}}\right)_{,i}-\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right)^{\cdot}-\boldsymbol{Q}_\theta=0 \tag{12.2}$$

$$J_4=D_{4\beta,\beta}+Q_{J_4}=\left(L+\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\dot{\boldsymbol{u}}+\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\dot{\boldsymbol{\theta}}\right)^{\cdot}-\left(\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\dot{\boldsymbol{u}}+\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,i}}\dot{\boldsymbol{\theta}}\right)_{,i}+\boldsymbol{Q}_u\dot{\boldsymbol{u}}+\boldsymbol{Q}_\theta\dot{\boldsymbol{\theta}}=0 \tag{12.3}$$

$$J_i=D_{i\beta,\beta}+Q_{Ji}=\left(\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\boldsymbol{u}_{,i}+\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\boldsymbol{\theta}_{,i}\right)^{\cdot}+\left(L\delta_{ij}-\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,j}}\boldsymbol{u}_{,i}-\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,j}}\boldsymbol{\theta}_{,i}\right)_{,j}+\boldsymbol{Q}_u\boldsymbol{u}_{,i}+\boldsymbol{Q}_\theta\boldsymbol{\theta}_{,i}=0 \tag{12.4}$$

$$\begin{aligned}M=M_{\alpha,\alpha}+Q_M=&\Big[Lt+\frac{\partial T}{\partial \dot{\boldsymbol{u}}}(\boldsymbol{u}+\boldsymbol{u}_{,j}x_j+\dot{\boldsymbol{u}}t)+\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}(\boldsymbol{\theta}+\boldsymbol{\theta}_{,j}+x_j+\dot{\boldsymbol{\theta}}t)\Big]^{\cdot}\\&+\Big[Lx_i-\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}(\boldsymbol{u}+\boldsymbol{u}_{,j}x_j+\dot{\boldsymbol{u}}t)-\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,i}}\cdot(\boldsymbol{\theta}+\boldsymbol{\theta}_{,j}x_j+\dot{\boldsymbol{\theta}}t)\Big]_{,i}+\boldsymbol{Q}_u(\boldsymbol{u}+\boldsymbol{u}_{,j}x_j+\dot{\boldsymbol{u}}t)\\&+\boldsymbol{Q}_\theta(\boldsymbol{\theta}+\boldsymbol{\theta}_{,j}x_j+\dot{\boldsymbol{\theta}}t)=0\end{aligned} \tag{12.5}$$

$$\begin{aligned}L_k=M_{k\alpha,\alpha}+Q_{L_k}=&\left[\left(\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\boldsymbol{u}_{,i}+\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\boldsymbol{\theta}_{,i}\right)\varepsilon_{ijk}x_j-\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\boldsymbol{u}\varepsilon_k\right]^{\cdot}\\&+\left[\left(L\delta_{li}-\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,l}}\boldsymbol{u}_{,i}-\boldsymbol{m}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,l}}\boldsymbol{\theta}_{,i}\right)\varepsilon_{ijk}x_j+\boldsymbol{\sigma}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,l}}\boldsymbol{u}\varepsilon_k\right]_{,l}\\&+\boldsymbol{Q}_u(\boldsymbol{u}_{,i}\varepsilon_{ijk}x_j-\boldsymbol{u}\varepsilon_k)+\boldsymbol{Q}_\theta\boldsymbol{Q}_{,i}\varepsilon_{ijk}x_j=0\end{aligned} \tag{12.6}$$

由定理 **2** 知，证明（12）式有二种等价的办法，一是（11）式必须满足广义 Killing 方

程组 (5.2)* 式，二是 (12) 式必须满足场方程 (5.1) 式. 这二种办法都进行计算，得到相同的结果如下(其中 (12.1) 是恒等式):

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\theta}}+\varepsilon_i\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}=0 \tag{12.2)'$$

$$\frac{\partial L}{\partial t}=0 \tag{12.3)'$$

$$\frac{\partial L}{\partial x_i}=0 \tag{12.4)'$$

$$\begin{aligned}\frac{\partial L}{\partial x_\alpha}x_\alpha+2\Big[2L&+\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\dot{\boldsymbol{u}}+\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\dot{\boldsymbol{\theta}}-\frac{\partial V}{\partial \boldsymbol{e}}\Big(\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\boldsymbol{u}_{,i}\\&+\frac{1}{2}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\theta}}\boldsymbol{\theta}\Big)-\frac{\partial V}{\partial \boldsymbol{\mu}}\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_{,i}}\boldsymbol{\theta}_{,i}\Big]=0\end{aligned} \tag{12.5)'$$

$$\begin{aligned}\varepsilon_{ijk}\Big(\frac{\partial L}{\partial x_i}x_j&-\frac{\partial V}{\partial \boldsymbol{u}_{,j}}\boldsymbol{u}_{,i}-\frac{\partial V}{\partial \boldsymbol{\theta}_{,j}}\boldsymbol{\theta}_{,i}\Big)\\&+\varepsilon_k\Big(\frac{\partial V}{\partial \boldsymbol{u}_{,l}}\boldsymbol{u}_{,l}-\frac{\partial T}{\partial \dot{\boldsymbol{u}}}\dot{\boldsymbol{u}}\Big)=0\end{aligned} \tag{12.6)'$$

我们给出的 L 是一般形式，$T(x_\alpha, \dot{\boldsymbol{u}}, \dot{\boldsymbol{\theta}})$，$V(x_\alpha, \boldsymbol{e}, \boldsymbol{\mu})$ 具有非均匀异性非线性性质，$\boldsymbol{e}(\boldsymbol{u}_{,i}, \boldsymbol{\theta})$，$\boldsymbol{\mu}(\boldsymbol{\theta}_{,i})$ 具有非线性性质. 由 (12) 式成立的条件 (12)′ 式再给以限制. 式 (12.2) 成立的条件是满足几何关系 (12.2)′ 式，此时 (12.2) 即为 (10) 式中的第二个场方程. 对于式 (12.3)，(12.4) $J_\alpha=0$ 成立的条件是 L 不依赖于第 α 维坐标 x_α. 式 (12.5) 成立的条件是 L 不依赖四个坐标量，T，V 为齐二次即线性的惯性、物性关系，$\boldsymbol{\mu}(\boldsymbol{\theta}_{,i})$ 为一次式，$\boldsymbol{e}=\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\boldsymbol{u}_{,i}+\frac{1}{2}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\theta}}\boldsymbol{\theta}$. (12.6) 式成立的条件是线性及满足 (12.6)′ 式所规定的包括关于空间坐标的同性性质.

注意 (12.2)′ 和 (12.5)′ 式的二种不同的几何关系限制条件. (12.2)′ 式是线性，在最简单的单个线、角位移变量情况下是微极线弹性介质的几何关系: $e_{ij}=u_{i,j}-\varepsilon_{ijk}\theta_k$. (12.5)′ 式是 $\boldsymbol{e}(\boldsymbol{u}_{,i}, \boldsymbol{\theta})$ 对 $\boldsymbol{u}_{,i}$ 一次，对 $\boldsymbol{\theta}$ 二次的几何关系

$$\boldsymbol{e}=\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{u}_{,i}}\boldsymbol{u}_{,i}+\frac{1}{2}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{\theta}}\boldsymbol{\theta} \tag{12.2)''$$

一方面看出各个守恒律的限制条件不一致. 某些守恒律的限制条件很少. 例如 $J_\alpha=0$，仅要求 L 不依赖于第 α 维坐标. 在应用某个守恒律时只受特定的限制条件，而可以扩大非受限制方面的物理内容.

另一方面是守恒律的完整性问题. 场方程 (10) 式对应二组守恒律与变换律. 一组是满足几何关系 (12.2)′ 式、均匀、线性、同性、守恒律是除了 (12.5) 式的 (12) 式中的五个，变换函数使(11)式中去掉与系数 $\boldsymbol{\nu}$ 有关的项. 一组是满足几何关系(12.2)″ 式、均匀、线性(不包括变量 $\boldsymbol{\theta}$)、同性，守恒律是除了(12.2)式的 (12) 式中的五个，变换函数是去掉 (11) 式中与系数 $\boldsymbol{\alpha}$ 有关的项.

从得到的二组几何关系式又证实了，通过普遍模型的守恒定律，有助于构造某类连续介质力学的基本方程.

用于微极线弹性介质，再看作保守力，则可与文献 [1] 得到的结果进行比较，二者的

(11.4) (11.5) 有所不同，我们作一讨论．因为 (11.4) 式在文献 [1] 的取法上与这里不同：$\boldsymbol{F}=\boldsymbol{a}$，即等号右边少了与系数 ν 有关的变换量中的 $-\boldsymbol{\theta}\nu$ 的项，对应文献 [1] 得到的守恒律 (12.5) 式中则少了 $\left(\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\boldsymbol{\theta}\right)^{\cdot}-\left(\boldsymbol{m}\frac{\partial\boldsymbol{\mu}}{\partial\boldsymbol{\theta}_{,i}}\boldsymbol{\theta}\right)_{,i}+Q_{\theta}\boldsymbol{\theta}$ 的项，守恒律成立条件应为：

$$\frac{\partial L}{\partial x_{\alpha}}x_{\alpha}+2\left(2L+\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\dot{\boldsymbol{u}}+\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\dot{\boldsymbol{\theta}}-\frac{\partial V}{\partial\boldsymbol{u}_{,i}}\boldsymbol{u}_{,i}-\frac{\partial V}{\partial\boldsymbol{\theta}_{,i}}\boldsymbol{\theta}_{,i}-\frac{\partial V}{\partial\boldsymbol{\theta}}\boldsymbol{\theta}\right)$$
$$-\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\dot{\boldsymbol{\theta}}+\frac{\partial V}{\partial\boldsymbol{\theta}_{,i}}\boldsymbol{\theta}_{,i}+2\frac{\partial V}{\partial\boldsymbol{\theta}}\boldsymbol{\theta}=0$$

除了均匀、线性条件外还须要满足 $-\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\dot{\boldsymbol{\theta}}+\frac{\partial V}{\partial\boldsymbol{\theta}_{,i}}\boldsymbol{\theta}_{,i}+2\frac{\partial V}{\partial\boldsymbol{\theta}}\boldsymbol{\theta}=0$

这一点是难以实现的．即是说文献 [1] 关于式 (11.4) 的取法及得到的 (12.5) 式是值得商榷的．按这里所取的 (11.4) 式及得到的 (12.5) 式，其成立条件 (12.5)′ 式，在物理上是可行的，其中几何关系 (12.2)″ 式已不属微极线弹性理论．因此，式 (12.5) 对微极线弹性理论是不成立的．

守恒律的积分型式，跳变条件，可类似于文献 [1] 得到，无特殊困难而不赘述．

最后给出断裂力学应用的初步说明．介质内空穴 S_c，S_Γ 为包围 S_c 的任意封闭曲面及其内所围域 V，n_i 为面元单位法线．则易证下列积分与 S_Γ 无关(其中 $\tilde{J}_4$ 尚为恒零值) 因而是断裂力学应用的基础：

$$\begin{aligned}\tilde{J}_{\alpha}&=\int_V(D_{\alpha4,4}+Q_{J_\alpha})dV+\int_{S_\Gamma}D_{\alpha j}n_jdS=\int_{S_c}L\delta_{\alpha i}n_idS\\ \tilde{M}&=\int_V(M_{4,4}+Q_M)dV+\int_{S_\Gamma}M_jn_jdS=\int_{S_c}Lx_in_idS\\ \tilde{L}_k&=\int_V(M_{k4,4}+Q_{L_k})dV+\int_{S_\Gamma}M_{kj}n_jdS=\int_{S_c}\varepsilon_{ijk}Lx_jn_idS\end{aligned}\tag{13}$$

参　考　文　献

[1] 戴天民，微极线性弹性动力学中的守恒定律和跳变条件，力学学报，3(1981)，271．

THE THEOREM OF CONSERVATION OF NON-CONSERVATIVE FIELD AND THE CONSERVATION LAWS FOR A CERTAIN CLASS OF CONTINUUM MECHANICS

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Abstract

The theorem of conservation of non-conservative field has been derived based on the differential variational principle and the conservation laws for a certain class of continuum mechanics are given in this paper. This work can be considered as an extenstion and complement of paper [1] and discussions are made about some results of that paper.