有强化弹塑性平面问题的一般渐近解

顾 求 林

（清 华 大 学）

提要　本文提供一求解有强化弹塑性平面问题（包括平面应力及不可压缩平面应变）的一般性渐近摄动解法，其中材料的强化规律用一幂级数表示．此方法适用于所有弹性解为已知的情况．由承受均匀内压的厚壁筒的示例可见本文的方法是可靠的．

一、引　　言

有强化弹塑性平面问题在工程上有很重要的意义，但是由于数学上的困难，这类问题能够精确求解者为数极少，行之有效的一般性近似解析解法也不多见．Ивлев 等人曾应用摄动法解决了大量弹塑性平面问题[1]，但他们的工作主要是研究理想塑性问题．对于有强化材料，他们仅在原理上讨论了不可压缩平面应变问题，而且在应用摄动法中，假设对应于小参数为零的初始状态是单向拉伸，最后得出很复杂的结果．显然这样的解法有很大的限制，其结果的正确程度也无从得知．古国纪同志与作者曾借采用幂级数强化规律，成功地应用摄动法求解了类似有强化弹塑性问题[2][3]．本文应用此法求得了有强化弹塑性平面问题在全量理论下的一般渐近解，其优点如下：

1. 解答过程中除了假设材料强化规律可用幂级数表达外，没有任何含糊不清之处，而实际计算表明，幂级数强化规律与实验曲线可以近似相符[3][4][5]．

2. 解答适用于任何弹性解为已知的平面问题，包括平面应力与平面应变（不可压缩）情况．结果简洁明瞭，计算切实可行．

3. 对于承受均匀内压的厚壁筒，按本文方法求得的三次近似解与精确解按小参数的三次展开式完全一致．这点说明本文解法是可靠的．

二、基 本 方 程

设材料的强化规律为

$$\bar{\varepsilon}_e = \frac{2(1+\nu)}{\bar{E}}\bar{\sigma}_e + \bar{\alpha}_2\bar{\sigma}_e^3 + \bar{\alpha}_4\bar{\sigma}_e^5 + \cdots$$

其中 $\bar{E}$ 及 ν 为材料弹性模量及横向变形系数；$\bar{\alpha}_2, \bar{\alpha}_4\cdots$ 为材料常数，可由拉伸曲线确定；$\bar{\varepsilon}_e$ 及 $\bar{\sigma}_e$ 为应变强度及应力强度．根据全量理论，平面弹塑性问题应满足下列方程：

$$\bar{\sigma}_x = \frac{\partial^2\bar{\Phi}}{\partial\bar{y}^2},\quad \bar{\sigma}_y = \frac{\partial^2\bar{\Phi}}{\partial\bar{x}^2},\quad \bar{\tau}_{xy} = -\frac{\partial^2\bar{\Phi}}{\partial\bar{x}\partial\bar{y}}$$

本文于 1980 年 9 月收到．

$$\frac{\partial^2\bar{\varepsilon}_x}{\partial\bar{y}^2}+\frac{\partial^2\bar{\varepsilon}_y}{\partial\bar{x}^2}-\frac{\partial^2\bar{\gamma}_{xy}}{\partial\bar{x}\partial\bar{y}}=0$$

$$\bar{\varepsilon}_x=\bar{\varepsilon}_x^e+\bar{\varepsilon}_x^p,\quad \bar{\varepsilon}_y=\bar{\varepsilon}_y^e+\bar{\varepsilon}_y^p,\quad \bar{\gamma}_{xy}=\bar{\gamma}_{xy}^e+\bar{\gamma}_{xy}^p$$

$$\bar{\varepsilon}_x^e=\frac{1}{\bar{E}_1}(\bar{\sigma}_x-\nu_1\bar{\sigma}_y)$$

$$\bar{\varepsilon}_y^e=\frac{1}{\bar{E}_1}(\bar{\sigma}_y-\nu_1\bar{\sigma}_x)$$

$$\bar{\gamma}_{xy}^e=\frac{2(1+\nu_1)}{\bar{E}_1}\bar{\tau}_{xy}$$

$$\left.\begin{aligned}&\bar{E}_1=\bar{E},\quad \nu_1=\nu\\&\bar{\varepsilon}_x^p=(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)\left(\bar{\sigma}_x-\frac{1}{2}\bar{\sigma}_y\right)\\&\bar{\varepsilon}_y^p=(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)\left(\bar{\sigma}_y-\frac{1}{2}\bar{\sigma}_x\right)\\&\bar{\gamma}_{xy}^p=3(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)\bar{\tau}_{xy}\\&\bar{\sigma}_e^2=\bar{\sigma}_x^2-\bar{\sigma}_x\bar{\sigma}_y+\bar{\sigma}_y^2+3\bar{\tau}_{xy}^2\end{aligned}\right\}\text{（平面应力）}$$

$$\left.\begin{aligned}&\bar{E}_1=\frac{4}{3}\bar{E},\quad \nu_1=1\\&\bar{\varepsilon}_x^p=\frac{3}{4}(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)(\bar{\sigma}_x-\bar{\sigma}_y)\\&\bar{\varepsilon}_y^p=\frac{3}{4}(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)(\bar{\sigma}_y-\bar{\sigma}_x)\\&\bar{\gamma}_{xy}^p=3(\bar{\alpha}_2\bar{\sigma}_e^2+\bar{\alpha}_4\bar{\sigma}_e^4+\cdots)\bar{\tau}_{xy}\\&\bar{\sigma}_e^2=\frac{3}{4}[(\bar{\sigma}_x-\bar{\sigma}_y)^2+4\bar{\tau}_{xy}^2]\end{aligned}\right\}\text{（不可压缩平面应变）}$$

其中 $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ 为应力分量；$\bar{\Phi}$ 为应力函数；$\bar{x}, \bar{y}$ 为坐标；$\bar{\varepsilon}_x, \bar{\varepsilon}_y, \bar{\gamma}_{xy}$ 为总应变分量；$\bar{\varepsilon}_x^e, \bar{\varepsilon}_y^e, \bar{\gamma}_{xy}^e$ 为弹性应变分量；$\bar{\varepsilon}_x^p, \bar{\varepsilon}_y^p, \bar{\gamma}_{xy}^p$ 为塑性应变分量；$\bar{E}_1$ 及 ν_1 为折算弹性模量及横向变形系数．此外，若给定边界上外加应力，则应满足边界条件：

$$l\bar{\sigma}_x|_{边}+m\bar{\tau}_{xy}|_{边}=\bar{\sigma}_0 f_x$$
$$l\bar{\tau}_{xy}|_{边}+m\bar{\sigma}_y|_{边}=\bar{\sigma}_0 f_y$$

其中 $\bar{\sigma}_x|_{边}, \bar{\sigma}_y|_{边}, \bar{\tau}_{xy}|_{边}$ 为应力分量的边界值；l, m 为边界外法线的方向余弦；f_x 及 f_y 为已知函数；$\bar{\sigma}_0$ 为外加边界应力参数．

为将诸方程无量纲化，采用下列变换：

$$\bar{\Phi}=\bar{\sigma}_s a^2\Phi,\quad \bar{x}=ax,\qquad \bar{y}=ay,$$
$$\bar{\sigma}_x=\bar{\sigma}_s\sigma_x,\quad \bar{\sigma}_y=\bar{\sigma}_s\sigma_y,\quad \bar{\tau}_{xy}=\bar{\sigma}_s\tau_{xy},$$
$$\bar{\sigma}_e=\bar{\sigma}_s\sigma_e,\quad \bar{\sigma}_0=\bar{\sigma}_s\sigma_0,$$
$$\bar{\varepsilon}_x=\frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_x,\quad \bar{\varepsilon}_y=\frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_y,\quad \bar{\gamma}_{xy}=\frac{\bar{\sigma}_s}{\bar{E}_1}\gamma_{xy},$$
$$\bar{\varepsilon}_x^e=\frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_x^e,\quad \bar{\varepsilon}_y^e=\frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_y^e,\quad \bar{\gamma}_{xy}^e=\frac{\bar{\sigma}_s}{\bar{E}_1}\gamma_{xy}^e,$$

$$\bar{\varepsilon}_x^p = \frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_x^p,\ \bar{\varepsilon}_y^p = \frac{\bar{\sigma}_s}{\bar{E}_1}\varepsilon_y^p,\ \bar{\gamma}_{xy}^p = \frac{\bar{\sigma}_s}{\bar{E}_1}\gamma_{xy}^p,$$

$$\bar{\varepsilon}_e = \frac{\bar{\sigma}_s}{\bar{E}}\varepsilon_e,\ \bar{\alpha}_2 = \frac{1}{\bar{E}\bar{\sigma}_s^2}\alpha_2,\ \bar{\alpha}_4 = \frac{1}{\bar{E}\bar{\sigma}_s^4}\alpha_4\cdots$$

其中 $\Phi, \sigma_x, \sigma_y, \tau_{xy}, \sigma_e, \sigma_0, \varepsilon_x, \varepsilon_y, \gamma_{xy}, \varepsilon_x^e, \varepsilon_y^e, \gamma_{xy}^e, \varepsilon_x^p, \varepsilon_y^p, \gamma_{xy}^p, \varepsilon_e, x, y, \alpha_2, \alpha_4\cdots$ 为相应的无量纲量；a 为某长度；$\bar{\sigma}_s$ 为材料的屈服极限．于是强化规律为

$$\varepsilon_e = \frac{2(1+\nu)}{3}\sigma_e + \alpha_2\sigma_e^3 + \alpha_4\sigma_e^5 + \cdots \tag{1}$$

基本方程为

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2},\quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} \tag{2}$$

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\partial y} = 0 \tag{3}$$

$$\varepsilon_x = \varepsilon_x^e + \varepsilon_x^p,\ \varepsilon_y = \varepsilon_y^e + \varepsilon_y^p,\ \gamma_{xy} = \gamma_{xy}^e + \gamma_{xy}^p \tag{4}$$

$$\varepsilon_x^e = \sigma_x - \nu_1\sigma_y,\ \varepsilon_y^e = \sigma_y - \nu_1\sigma_x,\ \gamma_{xy}^e = 2(1+\nu_1)\tau_{xy} \tag{5}$$

$$\left.\begin{aligned}\varepsilon_x^p &= (\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)\left(\sigma_x - \frac{1}{2}\sigma_y\right)\\ \varepsilon_y^p &= (\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)\left(\sigma_y - \frac{1}{2}\sigma_x\right)\\ \gamma_{xy}^p &= 3(\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)\tau_{xy}\\ \sigma_e^2 &= \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\end{aligned}\right\}\text{(平面应力)} \tag{6}$$

$$\left.\begin{aligned}\varepsilon_x^p &= (\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)(\sigma_x - \sigma_y)\\ \varepsilon_y^p &= (\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)(\sigma_y - \sigma_x)\\ \gamma_{xy}^p &= 4(\alpha_2\sigma_e^2 + \alpha_4\sigma_e^4 + \cdots)\tau_{xy}\\ \sigma_e^2 &= \frac{3}{4}[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]\end{aligned}\right\}\text{(不可压缩平面应变)} \tag{7}$$

边界条件为

$$\left.\begin{aligned}l\sigma_x|_{边} + m\tau_{xy}|_{边} &= \sigma_0 f_x\\ l\tau_{xy}|_{边} + m\sigma_y|_{边} &= \sigma_0 f_y\end{aligned}\right\} \tag{8}$$

将 (2), (4), (5) 式代入 (3) 式，得

$$\nabla^4\Phi + \frac{\partial^2\varepsilon_x^p}{\partial y^2} + \frac{\partial^2\varepsilon_y^p}{\partial x^2} - \frac{\partial^2\gamma_{xy}^p}{\partial x\partial y} = 0 \tag{9}$$

问题归结为求解方程组 (2), (6) 或 (7) 及 (9) 式，并满足边界条件 (8) 式．

三、渐 近 解

取 σ_0 为摄动小参数，即令

$$\Phi = \Phi_1\sigma_0 + \Phi_3\sigma_0^3 + \Phi_5\sigma_0^5 + \cdots \tag{10}$$

则相应地有

$$\left.\begin{aligned}\sigma_x &= \sigma_{x1}\sigma_0 + \sigma_{x3}\sigma_0^3 + \sigma_{x5}\sigma_0^5 + \cdots\\ \sigma_y &= \sigma_{y1}\sigma_0 + \sigma_{y3}\sigma_0^3 + \sigma_{y5}\sigma_0^5 + \cdots\\ \tau_{xy} &= \tau_{xy1}\sigma_0 + \tau_{xy3}\sigma_0^3 + \tau_{xy5}\sigma_0^5 + \cdots\end{aligned}\right\} \tag{11}$$

而其中

$$\sigma_{xi} = \frac{\partial^2\Phi_i}{\partial y^2},\quad \sigma_{yi} = \frac{\partial^2\Phi_i}{\partial x^2},\quad \tau_{xyi} = -\frac{\partial^2\Phi_i}{\partial x\partial y}\quad (i = 1, 3, 5\cdots) \tag{12}$$

代入(6)或(7)式得

$$\left.\begin{aligned}\varepsilon_x^p &= \varepsilon_{x3}^p\sigma_0^3 + \varepsilon_{x5}^p\sigma_0^5 + \cdots\\ \varepsilon_y^p &= \varepsilon_{y3}^p\sigma_0^3 + \varepsilon_{y5}^p\sigma_0^5 + \cdots\\ \gamma_{xy}^p &= \gamma_{xy3}^p\sigma_0^3 + \gamma_{xy5}^p\sigma_0^5 + \cdots\end{aligned}\right\} \tag{13}$$

其中

$$\left.\begin{aligned}\varepsilon_{x3}^p &= \alpha_2\sigma_{c1}^2\left(\sigma_{x1} - \frac{1}{2}\sigma_{y1}\right)\\ \varepsilon_{y3}^p &= \alpha_2\sigma_{c1}^2\left(\sigma_{y1} - \frac{1}{2}\sigma_{x1}\right)\\ \gamma_{xy3}^p &= 3\alpha_2\sigma_{c1}^2\tau_{xy}\\ \varepsilon_{x5}^p &= \alpha_2\sigma_{c1}^2\left(\sigma_{x3} - \frac{1}{2}\sigma_{y3}\right) + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)\left(\sigma_{x1} - \frac{1}{2}\sigma_{y1}\right)\\ \varepsilon_{y5}^p &= \alpha_2\sigma_{c1}^2\left(\sigma_{y3} - \frac{1}{2}\sigma_{x3}\right) + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)\left(\sigma_{y1} - \frac{1}{2}\sigma_{x1}\right)\\ \gamma_{xy5}^p &= 3[\alpha_2\sigma_{c1}^2\tau_{xy3} + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)\tau_{xy1}]\end{aligned}\right\}\text{（平面应力）} \tag{14}$$

$$\left.\begin{aligned}\varepsilon_{x3}^p &= \alpha_2\sigma_{c1}^2(\sigma_{x1} - \sigma_{y1})\\ \varepsilon_{y3}^p &= \alpha_2\sigma_{c1}^2(\sigma_{y1} - \sigma_{x1})\\ \gamma_{xy3}^p &= 3\alpha_2\sigma_{c1}^2\tau_{xy}\\ \varepsilon_{x5}^p &= \alpha_2\sigma_{c1}^2(\sigma_{x3} - \sigma_{y3}) + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)(\sigma_{x1} - \sigma_{y1})\\ \varepsilon_{y5}^p &= \alpha_2\sigma_{c1}^2(\sigma_{y3} - \sigma_{x3}) + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)(\sigma_{y1} - \sigma_{x1})\\ \gamma_{xy5}^p &= 3[\alpha_2\sigma_{c1}^2\tau_{xy3} + (\alpha_2\sigma_{13} + \alpha_4\sigma_{c1}^4)\tau_{xy1}\end{aligned}\right\}\text{（不可压缩平面应变）} \tag{15}$$

其中

$$\left.\begin{aligned}\sigma_{c1}^2 &= \sigma_{x1}^2 - \sigma_{x1}\sigma_{y1} + \sigma_{y1}^2 + 3\tau_{xy1}^2\\ \sigma_{13} &= 2\sigma_{x1}\sigma_{x3} - \sigma_{x1}\sigma_{y3} - \sigma_{x3}\sigma_{y1} + 2\sigma_{y1}\sigma_{y3} + 6\tau_{xy1}\tau_{xy3}\end{aligned}\right\}\text{（平面应力）} \tag{16}$$

$$\left.\begin{aligned}\sigma_{c1}^2 &= \frac{3}{4}[(\sigma_{x1} - \sigma_{y1})^2 + 4\tau_{xy1}^2]\\ \sigma_{13} &= \frac{3}{2}(\sigma_{x1}\sigma_{x3} - \sigma_{x1}\sigma_{y3} - \sigma_{x3}\sigma_{y1} + \sigma_{y1}\sigma_{y3} + 8\tau_{xy1}\tau_{xy3})\end{aligned}\right\}\text{（不可压缩平面应变）} \tag{17}$$

将(10)及(13)式代入(9)及(8)式，比较 $\sigma_0, \sigma_0^3, \sigma_0^5\cdots$ 项的系数，即得逐次近似解.

1. 第一次近似——Φ_1 应满足的方程为

$$\nabla^4\Phi_1 = 0$$

边界条件为

$$l\sigma_{x1}|_{边} + m\tau_{xy1}|_{边} = f_x$$
$$l\tau_{xy1}|_{边} + m\sigma_{y1}|_{边} = f_y$$

此即弹性问题，其解答 σ_{x1}，σ_{y1}，τ_{xy1} 可认为已知.

2. 第二次近似——Φ_3 应满足的方程为

$$\nabla^4\Phi_3 = F_3 = -\frac{\partial^2\varepsilon^p_{x3}}{\partial y^2} - \frac{\partial^2\varepsilon^p_{y3}}{\partial x^2} + \frac{\partial^2\gamma^p_{xy3}}{\partial x\partial y} \tag{18}$$

边界条件为

$$\left.\begin{aligned} l\sigma_{x3}|_{边} + m\tau_{xy3}|_{边} = 0 \\ l\tau_{xy3}|_{边} + m\sigma_{y3}|_{边} = 0 \end{aligned}\right\} \tag{19}$$

因 σ_{x1}，σ_{y1}，τ_{xy1} 为已知，所以由 (14)—(17) 式即可求得 F_3. 设

$$\Phi_3 = \Phi_3^* + \Phi_3^0 \tag{20}$$

其中 Φ_3^* 及 Φ_3^0 分别是 (18) 式的特解及齐次解，则应力分量为

$$\sigma_{x3} = \sigma^*_{x3} + \sigma^0_{x3},\quad \sigma_{y3} = \sigma^*_{y3} + \sigma^0_{y3},\quad \tau_{xy3} = \tau^*_{xy3} + \tau^0_{xy3} \tag{21}$$

其中

$$\left.\begin{aligned} \sigma^*_{x3} = \frac{\partial^2\Phi_3^*}{\partial y^2},\quad \sigma^*_{y3} = \frac{\partial^2\Phi_3^*}{\partial x^2},\quad \tau^*_{xy3} = -\frac{\partial^2\Phi_3^*}{\partial x\partial y} \\ \sigma^0_{x3} = \frac{\partial^2\Phi_3^0}{\partial y^2},\quad \sigma^0_{y3} = \frac{\partial^2\Phi_3^0}{\partial x^2},\quad \tau^0_{xy3} = -\frac{\partial^2\Phi_3^0}{\partial x\partial y} \end{aligned}\right\} \tag{22}$$

应用复变函数运算，可求得特解 Φ_3^* 为

$$\Phi_3^* = \frac{1}{16}\iiiint F_3 dz dz d\bar{z} d\bar{z} \tag{23}$$

其中

$$z = x + iy,\quad \bar{z} = x - iy \tag{24}$$

再求齐次解 Φ_3^0，它应满足的方程为

$$\nabla^4\Phi_3^0 = 0 \tag{25}$$

由 (19) 及 (21) 式可导出其边界条件如下：

$$\left.\begin{aligned} l\sigma^0_{x3}|_{边} + m\tau^0_{xy3}|_{边} = -l\sigma^*_{x3}|_{边} - m\tau^*_{xy3}|_{边} \\ l\tau^0_{xy3}|_{边} + m\sigma^0_{y3}|_{边} = -l\tau^*_{xy3}|_{边} - m\sigma^*_{y3}|_{边} \end{aligned}\right\} \tag{26}$$

因 $\sigma^*_{x3}|_{边}$，$\sigma^*_{y3}|_{边}$，$\tau^*_{xy3}|_{边}$ 可由 (23) 及 (22) 式求得，所以确定齐次解 Φ_3^0 相当于求解已知边界条件的弹性问题.

3. 第三次近似——Φ_5 应满足的方程为

$$\nabla^4\Phi_5 = F_5 = -\frac{\partial^2\varepsilon^p_{x5}}{\partial y^2} - \frac{\partial^2\varepsilon^p_{y5}}{\partial x^2} + \frac{\partial^2\gamma^p_{xy5}}{\partial x\partial y}$$

边界条件为

$$l\sigma_{x5}|_{边} + m\tau_{xy5}|_{边} = 0$$
$$l\tau_{xy5}|_{边} + m\sigma_{y5}|_{边} = 0$$

容易看到，其解法与第二次近似相同.

以此类推，可以得到所需要的以下各次近似解.

四、例　　题

今以承受均匀内压的厚壁筒为例，说明上述方法的应用，并将渐近解与精确解作比较．设厚壁筒的内半径为 a，外径与内径之比为 k，内压强为 p．采用 $\sigma_0 = p/\bar{\sigma}_s$ 及无量纲极坐标 r, θ．考虑为不可压缩平面应变问题，则按以上计算可得下列结果．

1. 第一次近似——即为弹性解：

$$\sigma_{r1} = (1 - k^2 r^{-2})(k^2 - 1)^{-1}$$
$$\sigma_{t1} = (1 + k^2 r^{-2})(k^2 - 1)^{-1}$$

其中 σ_r 及 σ_t 为径向及切向正应力．

2. 第二次近似——由 σ_{r1} 及 σ_{t1} 按（15）及（18）式，求得

$$F_3 = -144\alpha_2 k^6 r^{-8}(k^2 - 1)^{-3}$$

则特解 Φ_3^* 及相应的应力分量 σ_{r3}^*，σ_{t3}^* 为

$$\Phi_3^* = \frac{1}{16}\iiiint F_3 dz dz d\bar{z} d\bar{z} = -\frac{1}{4}\alpha_2 k^6 r^{-4}(k^2 - 1)^{-3}$$
$$\sigma_{r3}^* = \alpha_2 k^6 r^{-6}(k^2 - 1)^{-3}$$
$$\sigma_{t3}^* = -5\alpha_2 k^6 r^{-6}(k^2 - 1)^{-3}$$

齐次解的应力分量(即承受均匀内、外压的厚壁筒的弹性解)为

$$\sigma_{r3}^0 = A_3 + B_3 r^{-2}$$
$$\sigma_{t3}^0 = A_3 - B_3 r^{-2}$$

其中 A_3 及 B_3 为待定常数．根据（26）式可得齐次解的边界条件为

$$\sigma_{r3}^0|_{r=1} = -\sigma_{r3}^*|_{r=1}, \quad \sigma_{r3}^0|_{r=k} = -\sigma_{r3}^*|_{r=k}$$

由此求得

$$A_3 = \alpha_2 k^2(k^2 + 1)(k^2 - 1)^{-3}$$
$$B_3 = -\alpha_2 k^2(k^4 + k^2 + 1)(k^2 - 1)^{-3}$$

3. 第三次近似——由 σ_{r1}，σ_{t1}，σ_{r3} 及 σ_{t3} 按（15）及（18）式，求得

$$F_5 = 144k^6[-3\alpha_2^2(k^4 + k^2 + 1)r^{-8} + 10(3\alpha_2^2 + \alpha_4)k^4 r^{-12}](k^2 - 1)^{-5}$$

则特解及相应的应力分量为

$$\Phi_5^* = \frac{1}{40}k^2[-30\alpha_2^2(k^4 + k^2 + 1)r^{-4} + 9(3\alpha_2^2 - \alpha_4)k^4 r^{-8}](k^2 - 1)^{-5}$$

$$\sigma_{r5}^* = \frac{3}{5}k^6[5\alpha_2^2(k^4 + k^2 + 1)r^{-6} - 3(3\alpha_2^2 - \alpha_4)k^4 r^{-10}](k^2 - 1)^{-5}$$

$$\sigma_{t5}^* = \frac{3}{5}k^6[-25\alpha_2^2(k^4 + k^2 + 1)r^{-6} + 27(3\alpha_2^2 - \alpha_4)k^4 r^{-10}](k^2 - 1)^{-5}$$

齐次解为

$$\sigma_{r5}^0 = A_5 + B_5 r^{-2}$$
$$\sigma_{t5}^0 = A_5 - B_5 r^{-2}$$

由边界条件确定 A_5 及 B_5 为

$$A_5 = \frac{3}{5}k^2[5\alpha_2^2(k^4 + k^2 + 1)(k^4 - 1) - 3(3\alpha_2^2 - \alpha_4)(k^8 - 1)](k^2 - 1)^{-6}$$

$$B_5=\frac{3}{5}k^2[-5\alpha_2^2(k^4+k^2+1)(k^6-1)+3(3\alpha_2^2-\alpha_4)(k^{10}-1)](k^2-1)^{-6}$$

最后结果为

$$\sigma_r=\sigma_{r1}\sigma_0+\sigma_{r3}\sigma_0^3+\sigma_{r5}\sigma_0^5+\cdots$$
$$\sigma_t=\sigma_{t1}\sigma_0+\sigma_{t3}\sigma_0^3+\sigma_{t5}\sigma_0^5+\cdots$$

此问题的精确解为已知[6],其无量纲形式如下:

$$\sigma_0=\frac{1}{\sqrt{3}}\int_{\varepsilon_{ek}}^{\varepsilon_{e1}}\frac{\sigma_e}{\varepsilon_e}d\varepsilon_e$$

$$\sigma_r=-\frac{1}{\sqrt{3}}\int_{\varepsilon_e}^{\varepsilon_{ek}}\frac{\sigma_e}{\varepsilon_e}d\varepsilon_e$$

$$\sigma_t=\frac{2}{\sqrt{3}}\sigma_e+\sigma_r$$

$$\varepsilon_{e1}=\varepsilon_e r^2=\varepsilon_{ek}k^2$$

其中 ε_{e1} 及 ε_{ek} 为厚壁筒内及外壁处的应变强度. 为了与渐近解作比较，先将强化规律(1)进行变换,并采用 $\nu=\frac{1}{2}$, 可得

$$\sigma_e=\varepsilon_e-\alpha_2\varepsilon_e^3+(3\alpha_2^2-\alpha_4)\varepsilon_e^5+\cdots$$

将此式代入精确解,积分后得

$$\sigma_0=\frac{1}{\sqrt{3}}\left[(k^2-1)\varepsilon_{ek}-\frac{1}{3}\alpha_2(k^6-1)\varepsilon_{ek}^3+\frac{1}{5}(3\alpha_2^2-\alpha_4)(k^{10}-1)\varepsilon_{ek}^5+\cdots\right]$$

$$\sigma_r=\frac{1}{\sqrt{3}}\left[(1-k^2r^{-2})\varepsilon_{ek}-\frac{1}{3}\alpha_2(1-k^6r^{-6})\varepsilon_{ek}^3+\frac{1}{5}(3\alpha_2^2-\alpha_4)(1-k^{10}r^{-10})\varepsilon_{ek}^5+\cdots\right]$$

$$\sigma_t=\frac{2}{\sqrt{3}}[k^2r^{-2}\varepsilon_{ek}-\alpha_2k^6r^{-6}\varepsilon_{ek}^3+(3\alpha_2^2-\alpha_4)k^{10}r^{-10}\varepsilon_{ek}^5+\cdots]+\sigma_r$$

将 σ_0 的表达式进行变换,可得

$$\varepsilon_{ek}=\sqrt{3}(k^2-1)^{-1}\sigma_0+\sqrt{3}\alpha_2(k^6-1)(k^2-1)^{-4}\sigma_0^3+\frac{3\sqrt{3}}{5}[5\alpha_2^2(k^6-1)^2(k^2-1)^{-7}-3(3\alpha_2^2-\alpha_4)(k^{10}-1)(k^2-1)^{-6}]\sigma_0^5+\cdots$$

将此式代入 σ_r 及 σ_t 的表达式,可得 σ_r 及 σ_t 按 σ_0 的展开式，其前三项与第一、二、三次近似完全相符.

五、结　语

由以上讨论可见，本文提出的一般渐近解法适用于所有弹性解为已知的有强化弹塑性平面问题. 由文献[3]可知,当最大应变不太大时,此种渐近解与实验结果很接近.作者将此方法应用于一个有强化弹塑性含椭圆孔拉板问题,并进行了实验验证，结果表明，当

最大应变在等于两倍弹性极限应变的范围内，理论与实验符合较好，其详情将另文发表.

参　考　文　献

[1] Ивлев, Д. Д., Ершов, Л. В., Метод Возмущений в Теории Упругопластического Тела, Издательство «Наука», 1978.

[2] 古国纪、顾璆琳，弹塑性圆板大挠度问题，力学学报，**2**, 3(1958), 232.

[3] 顾求林，弹塑性梁的挠度及轴向位移，力学学报，**7**, 2(1964), 149.

[4] Trifan, D., On the Plastic Bending of Circular Plates, *Quart. of Applied Math.*, **6** (1948).

[5] Гениев, Г. А., Исследовании по Вопросам Строительной Механики и Теории Пластичности, ЦНИПС (1956).

[6] 赵祖武，塑性理论基础，人民教育出版社 (1963).

A GENERAL ASYMPTOTIC ANALYTICAL SOLUTION OF PLANE PROBLEM OF ELASTO-PLASTICITY WITH STRAIN-HARDENING

Gu Qiulin

(*Tsing-Hua University*)

Abstract

In this paper a general asymptotic analytical solution of plane problem (including plane stress and plane strain) of elasto-plasticity with strain-hardening by perturbation method is suggested while the stress-strain relation of the material is expressed by a power series. The method is applicable to all cases with known elastic solution. The example of a thick-wall cylinder with uniform internal pressure shows that the method suggested in this paper is reliable.