涡量脉动相似结构和圆形涡旋流速分布

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1. 柱坐标中的湍流脉动方程

我们取柱坐标 (r, ϕ, z)，用 (U_r, U_ϕ, U_z) 代表瞬时速度. 我们假定流体为粘性不可压缩的. ρ 和 μ 都是常数. 我们把瞬时速度分为平均速度和涨落速度之和. 瞬时压强为平均压强 $\bar{p}$ 和压强涨落 ω 之和. 即

$$U_r = \bar{U}_r + u_r,\ U_\phi = \bar{U}_\phi + u_\phi,\ U_z = \bar{U}_z + u_z,\ p = \bar{p} + \omega$$

把 Navier-Stokes 方程和相应的 Reynolds 方程相减，就得到涨落速度所满足的方程组.

$$\rho\left(\frac{\partial u_r}{\partial t} + \bar{U}_r\frac{\partial u_r}{\partial r} + \frac{\bar{U}_\phi}{r}\frac{\partial u_r}{\partial \phi} + \bar{U}_z\frac{\partial u_r}{\partial z} + u_r\frac{\partial \bar{U}_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial \bar{U}_r}{\partial \phi} + u_z\frac{\partial \bar{U}_r}{\partial z}\right.$$
$$\left.+ u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2 + 2\bar{U}_\phi u_\phi}{r}\right)$$
$$= -\frac{\partial \omega}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right) + \frac{\rho}{r}\frac{\partial}{\partial r}(r\overline{u_r^2})$$
$$+ \rho\frac{\partial}{r\partial \phi}\overline{u_r u_\phi} + \rho\frac{\partial}{\partial z}\overline{u_r u_z} - \rho\frac{\overline{u_\phi^2}}{r} \tag{1}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t} + \bar{U}_r\frac{\partial u_\phi}{\partial r} + \frac{\bar{U}_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \bar{U}_z\frac{\partial u_\phi}{\partial z} + u_r\frac{\partial \bar{U}_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial \bar{U}_\phi}{\partial \phi} + u_z\frac{\partial \bar{U}_\phi}{\partial z}\right.$$
$$\left.+ u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi + \bar{U}_r u_\phi + u_r\bar{U}_\phi}{r}\right)$$
$$= -\frac{\partial \omega}{r\partial \phi} + \mu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right) + \rho\frac{1}{r}\frac{\partial}{\partial \phi}\overline{u_\phi^2}$$
$$+ \rho\frac{\partial}{\partial r}\overline{u_\phi u_r} + \rho\frac{\partial}{\partial z}\overline{u_\phi u_z} + 2\rho\frac{\overline{u_\phi u_r}}{r} \tag{2}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + \bar{U}_r\frac{\partial u_z}{\partial r} + \bar{U}_\phi\frac{\partial u_z}{r\partial \phi} + \bar{U}_z\frac{\partial u_z}{\partial z} + u_r\frac{\partial \bar{U}_z}{\partial r} + u_\phi\frac{\partial \bar{U}_z}{r\partial \phi} + u_z\frac{\partial \bar{U}_z}{\partial z}\right.$$
$$\left.+ u_r\frac{\partial u_z}{\partial r} + u_\phi\frac{\partial u_z}{r\partial \phi} + u_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial \omega}{\partial z} + \mu\nabla^2 u_z + \rho\frac{\partial \overline{u_z^2}}{\partial z} + \frac{\rho}{r}\frac{\partial}{\partial r}(r\overline{u_r u_z}) + \frac{\rho}{r}\frac{\partial}{\partial \phi}\overline{u_\phi u_z} \tag{3}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0 \tag{4}$$

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消去压强涨落 ω，得到涡量涨落方程(方程式从略).

2. 圆形涡旋的涡量涨落方程

我们考虑一个运动着的圆形涡旋，它的中心的速度和加速度各自为 v_i 和 a_i. 我们考虑一个运动坐标系，它的原点和圆形涡旋的中心重合. 由于加速度 a_i 在相减的过程中消去，这样速度涨落方程和静止坐标系中的速度涨落方程完全一样. 同样涡量方程也和静止涡旋在静止坐标系中的涡量方程一样.

对于圆形涡旋，平均流速只有一个分量 $\overline{U}_\phi$，而 $\overline{U}_\phi$ 仅仅是 r 的函数. 同样其他平均量也只是 r 的函数. 我们略去 $\overline{U}_\phi$ 的下标 ϕ 及平均号"—".

我们来考虑空间一点 (r_0, ϕ_0, z_0) 附近的速度涨落[1]. 令 $\xi_1 = (r - r_0)/\Lambda$, $\xi_2 = r(\phi - \phi_0)/\Lambda$, $\xi_3 = (z - z_0)/\Lambda$，其中 Λ 为脉动的特征长度. 令 $u_r = q\phi_1(\xi_i)$, $u_\phi = q\phi_2(\xi_i)$, $u_z = q\phi_3(\xi_i)$，式中 q 为特征速度. 如果认为涨落速度局部地随时间的变化不快，我们就可以把涨落速度对时间的偏导数项略去，于是涡量方程简化为

$$
\begin{aligned}
&\frac{qrU}{\Lambda^2}\frac{\partial^2\phi_1}{\partial\xi_2^2} + \frac{q^2 r}{\Lambda^2}\left(\frac{\partial\phi_1}{\partial\xi_2}\frac{\partial\phi_1}{\partial\xi_1} + \phi_1\frac{\partial^2\phi_1}{\partial\xi_1\partial\xi_2} + \frac{\partial\phi_2}{\partial\xi_2}\frac{\partial\phi_1}{\partial\xi_2} + \phi_2\frac{\partial^2\phi_1}{\partial\xi_2^2} + \frac{\partial\phi_3}{\partial\xi_2}\frac{\partial\phi_1}{\partial\xi_3}\right.\\
&\quad\left. + \phi_3\frac{\partial^2\phi_1}{\partial\xi_2\partial\xi_3}\right) - 2\frac{q^2}{\Lambda}\phi_2\frac{\partial\phi_2}{\partial\xi_2} - 2\frac{Uq}{\Lambda}\frac{\partial\phi_2}{\partial\xi_2} - \left(\frac{dU}{dr}\frac{qr}{\Lambda}\frac{\partial\phi_2}{\partial\xi_1} + \frac{Urq}{\Lambda^2}\frac{\partial^2\phi_2}{\partial\xi_2\partial\xi_1}\right.\\
&\quad + q\phi_1\frac{dU}{dr} + \frac{rq}{\Lambda}\frac{dU}{dr}\frac{\partial\phi_1}{\partial\xi_1} + rq\frac{d^2U}{dr^2}\phi_1 + \frac{q^2}{\Lambda}\phi_1\frac{\partial\phi_2}{\partial\xi_1} + \frac{q^2 r}{\Lambda^2}\frac{\partial\phi_1}{\partial\xi_1}\frac{\partial\phi_2}{\partial\xi_1}\\
&\quad + \frac{q^2}{\Lambda}\phi_1\frac{\partial^2\phi_2}{\partial\xi_1^2} + \frac{q^2 r}{\Lambda^2}\frac{\partial\phi_1}{\partial\xi_1}\frac{\partial\phi_2}{\partial\xi_2} + \frac{q^2 r}{\Lambda^2}\frac{\partial^2\phi_2}{\partial\xi_1\partial\xi_2} + \frac{q^2}{\Lambda}\phi_3\frac{\partial\phi_2}{\partial\xi_3} + \frac{q^2 r}{\Lambda^2}\frac{\partial\phi_3}{\partial\xi_1}\frac{\partial\phi_2}{\partial\xi_3}\\
&\quad\left. + \frac{q^2 r}{\Lambda^2}\frac{\partial^2\phi_2}{\partial\xi_1\partial\xi_3} + \frac{q^2}{\Lambda}\frac{\partial\phi_1}{\partial\xi_1}\phi_2 + \frac{q^2}{\Lambda}\phi_1\frac{\partial\phi_2}{\partial\xi_1} + \frac{dU}{dr}q\phi_1 + \frac{Uq}{\Lambda}\frac{\partial\phi_1}{\partial\xi_1}\right)\\
&= \frac{r\nu q}{\Lambda^3}\left(\nabla_\xi^2\frac{\partial\phi_1}{\partial\xi_2} - \frac{\Lambda^2}{r^2}\frac{\partial\phi_1}{\partial\xi_2} - \frac{2\Lambda}{r}\frac{\partial^2\phi_2}{\partial\xi_2^2}\right) - \frac{\nu q}{\Lambda^2}\left(\nabla_\xi^2\phi_2 + \frac{r}{\Lambda}\frac{\partial}{\partial\xi_1}\nabla_\xi^2\phi_2 + \frac{\Lambda}{r^2}\phi_2\right.\\
&\quad\left. - \frac{\Lambda}{r}\frac{\partial\phi_2}{\partial\xi_1} + 2\frac{\partial^2\phi_1}{\partial\xi_1\partial\xi_2} - \frac{2\Lambda}{r}\frac{\partial\phi_1}{\partial\xi_2}\right) - 3\frac{d}{dr}\overline{u_\phi u_r} - r\frac{d^2}{dr^2}\overline{u_\phi u_r}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\frac{qU}{\Lambda^2}\frac{\partial^2\phi_1}{\partial\xi_2\partial\xi_3} + \frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_1}{\partial\xi_3}\frac{\partial\phi_1}{\partial\xi_1} + \phi_1\frac{\partial^2\phi_1}{\partial\xi_1\partial\xi_3} + \frac{\partial\phi_1}{\partial\xi_2}\frac{\partial\phi_2}{\partial\xi_3} + \frac{\partial^2\phi_1}{\partial\xi_2\partial\xi_3}\phi_2 + \frac{\partial\phi_3}{\partial\xi_3}\frac{\partial\phi_1}{\partial\xi_3}\right.\\
&\quad\left. + \phi_3\frac{\partial^2\phi_2}{\partial\xi_3^2}\right) - \frac{q^2}{r\Lambda}\phi_2\frac{\partial\phi_2}{\partial\xi_3} - \frac{2qU}{r\Lambda}\frac{\partial\phi_2}{\partial\xi_3} - \left(\frac{dU}{dr}\frac{q}{\Lambda}\frac{\partial\phi_3}{\partial\xi_2} + \frac{Uq}{\Lambda^2}\frac{\partial^2\phi_3}{\partial\xi_2\partial\xi_1}\right.\\
&\quad\left. - \frac{Uq}{r\Lambda}\frac{\partial\phi_3}{\partial\xi_2}\right) - \frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_1}{\partial\xi_1}\frac{\partial\phi_3}{\partial\xi_1} + \frac{\partial^2\phi_3}{\partial\xi_1^2}\phi_1 + \frac{\partial\phi_2}{\partial\xi_1}\frac{\partial\phi_3}{\partial\xi_2} - \frac{\Lambda}{r}\phi_2\frac{\partial\phi_3}{\partial\xi_2}\right.\\
&\quad\left. + \phi_2\frac{\partial\phi_3}{\partial\xi_1\partial\xi_2} + \frac{\partial\phi_3}{\partial\xi_1}\frac{\partial\phi_3}{\partial\xi_3} + \phi_3\frac{\partial^2\phi_3}{\partial\xi_1\partial\xi_2}\right)\\
&= \nu\frac{q}{\Lambda^3}\left(\nabla_\xi^2\frac{\partial\phi_1}{\partial\xi_3} - \frac{\Lambda^2}{r^2}\frac{\partial\phi_1}{\partial\xi_3} - \frac{2\Lambda}{r}\frac{\partial^2\phi_2}{\partial\xi_2\partial\xi_3}\right) - \nu\frac{q}{\Lambda^3}\frac{\partial}{\partial\xi_1}\nabla_\xi^2\phi_3
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\frac{Urq}{\Lambda^2}\frac{\partial^2\phi_2}{\partial\xi_2\partial\xi_3} + r\frac{dU}{dr}\frac{q}{\Lambda}\frac{\partial\phi_1}{\partial\xi_3} + \frac{rq^2}{\Lambda^2}\left(\frac{\partial\phi_1}{\partial\xi_3}\frac{\partial\phi_2}{\partial\xi_1} + \phi_1\frac{\partial^2\phi_2}{\partial\xi_1\partial\xi_3} + \frac{\partial\phi_2}{\partial\xi_3}\frac{\partial\phi_2}{\partial\xi_2}\right.\\
&\quad\left. + \phi_2\frac{\partial^2\phi_2}{\partial\xi_2\partial\xi_3} + \frac{\partial\phi_3}{\partial\xi_3}\frac{\partial\phi_2}{\partial\xi_3} + \phi_3\frac{\partial^2\phi_2}{\partial\xi_3^2}\right) + \frac{q^2}{\Lambda}\phi_1\frac{\partial\phi_2}{\partial\xi_3} + \frac{q^2}{\Lambda}\phi_2\frac{\partial\phi_1}{\partial\xi_3} + \frac{qU}{\Lambda}\frac{\partial\phi_1}{\partial\xi_3}
\end{aligned}
$$

$$
-\left(\frac{qUr}{\Lambda^2}\frac{\partial^2\phi_3}{\partial\xi_2^2}+\frac{q^2r}{\Lambda^2}\frac{\partial\phi_1}{\partial\xi_2}\frac{\partial\phi_3}{\partial\xi_1}+\frac{q^2r}{\Lambda^2}\phi_1\frac{\partial^2\phi_3}{\partial\xi_1\partial\xi_2}+\frac{q^2r}{\Lambda^2}\frac{\partial\phi_2}{\partial\xi_2}\frac{\partial\phi_3}{\partial\xi_2}\right.
$$

$$
\left.+\frac{q^2r}{\Lambda^2}\phi_2\frac{\partial^2\phi_3}{\partial\xi_2^2}+\frac{q^2r}{\Lambda^2}\frac{\partial\phi_3}{\partial\xi_2}\frac{\partial\phi_3}{\partial\xi_3}+\frac{q^2r}{\Lambda^2}\phi_3\frac{\partial^2\phi_3}{\partial\xi_2\partial\xi_3}\right)
$$

$$
=\frac{\nu qr}{\Lambda^3}\left(\nabla_\xi^2\frac{\partial\phi_2}{\partial\xi_3}-\frac{\Lambda^2}{r^2}\frac{\partial\phi_2}{\partial\xi_3}+2\frac{\Lambda}{r}\frac{\partial^2\phi_1}{\partial\xi_2\partial\xi_3}\right)-\nu\frac{qr}{\Lambda^3}\nabla_\xi^2\phi_3 \tag{7}
$$

3. 相似性解和速度分布

当 R_A 很大时，粘性各项都可以略去，我们得到湍流脉动速度相似条件

$$
\left[\frac{qrU}{\Lambda^2}\right]\Big/\left[\frac{q^2r}{\Lambda^2}\right]=\frac{U}{q}=\frac{1}{c_1}\qquad q=c_1U \tag{8}
$$

$$
\left[\frac{q^2r}{\Lambda^2}\right]\Big/\left[\frac{q^2}{\Lambda}\right]=\frac{r}{\Lambda}=\frac{1}{c_2}\qquad \Lambda=c_2r \tag{9}
$$

$$
\left[\frac{dU}{dr}\frac{qr}{\Lambda}\right]\Big/\left[\frac{Uq}{\Lambda}\right]=\frac{dU}{dr}\frac{r}{U}=n\qquad \log U=n\log r+c'
$$

即

$$
U=kr^n \tag{10}
$$

它的适用范围有两个区域，一个是最大速度圆以外，一个是最大速度圆以内. 在最大速度圆附近因为 $dU/dr\simeq 0$，刚才的论断就不成立.

参　考　文　献

[1] 周培源，力学学报，3，4(1959)，281—296.

SIMILARITY OF THE STRUCTURE OF VORTICITY FLUCTUATION AND THE VELOCITY PROFILE OF CIRCULAR VORTEX

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Abstract

This article deals with the velocity profile of circular vortex according to the similarity of the structure of vorticity fluctuation. We divide the whole region into two parts——internal part and external part. For either part we obtain an average velocity profile. Where the two parts join, $\frac{du}{dr}\sim 0$. Then the above results do not apply.