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关于弹性扁壳边界补充条件問題*

裴 文 瑾
(建筑工程部西北工业建筑设计院)

文献[1]曾提出了确定四边简支矩形底扁壳边界应力函数的计算公式,此公式实为文献[2]所称的四边简支情形的补充条件. 文献[2]在提出扁壳的广义变分原理的同时, 利用此原理解决了许多扁壳边界问题,导出了比较广泛的扁壳边界补充条件,其结果我们曾在实际工作中有效地应用过. 现在本文提出一个求扁壳边界补充条件的结构力学方法[1],此法简明、直观,而且适用于各种边界情形.

1. 关于壳体的全部边界条件(包括补充条件), 必须把壳体与支承它的边缘构件及其他有关支承一同考虑. 例如四边简支建筑扁壳, 壳体本身通常通过边缘构件及四角点处的柱子与地基或其他结构联系,壳体的水平刚度一般比四角点处柱子的水平刚度大得多,故壳体应是后面附表的情形 1; 只有当支承壳体的柱子或其他结构的水平刚度甚大,以致能保证(或几乎能保证)壳体四角点无相对位移时, 才能是后面附表的情形 2. 这两种情况虽然都是四边简支,但补充条件是不同的. 也就是说,由于客观上存在着壳体及支承它的结构水平刚度的差异,所以就决定了在确定扁壳边界条件时,必须同时考虑壳体及边缘构件的支承情况. 因此,文献[1,2]关于四边简支情况的结果, 使用时必须符合附表情形 2 的支承情况.

具有一定自由度的结构系统, 可以用虚位移原理毫无遗漏地找出作用于此系统上诸力的关系(独立的), 这些关系的个数与自由度数目相同. 支承壳体的边缘构件是一个结构系统,壳体除角点(包括边界条件突变处)外的边界条件,常常不能完全反映边缘构件上力的全部关系,这种考虑边缘构件自由度而得到作用于其上的力(与壳体连接及不与壳体连接的所有力)的关系, 称为力的补充条件(还可写成考虑边缘构件变形位能的更一般的变分式,因后面未用到,故略去).

边缘构件的支承有时是超静定的,它们通过边缘构件与壳体联系,必须考虑壳体的变形方能确定这些超静定系统的內力, 由考虑壳体变形而补充的条件, 称为变形的补充条件.

以上二种补充条件构成了补充条件的全部,由此可知, 在考虑具体问题时, 先考虑边缘构件的自由度及超靜定性,求得补充条件的数目,便可避免补充条件遗漏情况的发生.

由于整个壳体结构的刚性位移并不影响內力分布, 可设壳体底平面內至少有三根不交于一点的连杆与之连系. 将能表示各边界两端无相对移动的边界 (例如固定、简支等) 看成连杆结构系统, 用结构力学方法即可求得其自由度数 n_1 (节点数目的两倍减去连杆数目)及与整个壳体结构连系的外部连杆超过三个的数目 n_2. 这时,补充条件的个数为

* 1965 年 1 月 5 日收到.
1) 本文所采用的符号及壳体內力方向与文献[2]及常见资料相同.

$$n = n_1 + n_2. \tag{1}$$

例 1　n 边简支(为明确起见,暂设为七边),有三根外连杆.

将边缘构件看成七根杆子的连杆系统,其自由度数 $n_1 = 14 - 10 = 4$, n_2 为 0, 故 $n = 4 + 0 = 4$. 这 4 个补充条件可由每一角点连杆为水平铰接之条件用虚位移原理求出各边之剪力合力 S_{12}—S_{71} 的任意四个独立关系式而得. 如果多求了(超过 4 个), 必是自然满足的, 故是无用的. 需注意的是, 每一个补充条件只能从每一角点为铰接的特性出发, 而不能用 7 个力 S_{12}—S_{71} 的总体平衡关系,因为总体平衡关系已包含在壳体的基本方程中了.

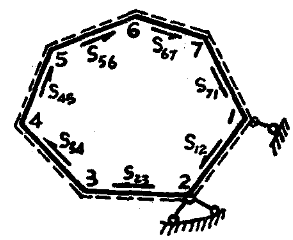


图　1

例 2　前例中有两个边换成一个自由边和一个固定边.

边界 3–4 两端相对位移不能由自由边限制,故不能看成是连杆. 7 个节点共有自由度 14 个,连杆有 9 根,故 $n_1 = 14 - 9 = 5$. 又 $n_2 = 0$, 故 $n = 5$. 但由壳体简支边法向应力为零,可知边界 2–3 绕点 2 无限制,即能表示节点 2 为铰接的特性,故有一个补充条件是自然满足的,即只需求 4 个补充条件就行了.

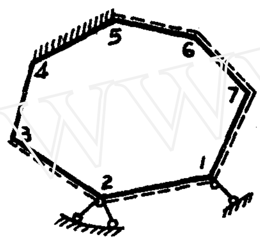


图　2

例 3　(后面附表情形 53).

因除角点外边界上无边缘构件,故无自由度,即 $n_1 = 0$, 又 $n_2 = 3$, 故 $n = 3$.

2. 现在叙述补充条件的求法. 这个方法对任意形状壳底均适合,但为简便及符合绝大多数实际问题需要起见,着重就矩形底情形进行了推导和编制了补充条件表.

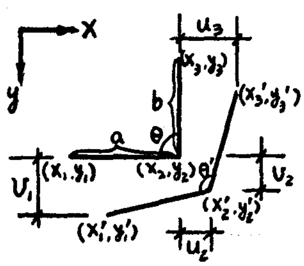


图　3

直角坐标系中原有位置关系为直角关系的三点 (x_1, y_1), (x_2, y_2), (x_3, y_3), 此三点产生位移后之位置为 (x_1', y_1'), (x_2', y_2'), (x_3', y_3'), 如果位移与原来三点相互间的尺度相比甚为微小,则

$$\theta' - \theta = \frac{v_1 - v_2}{a} + \frac{u_3 - u_2}{b}. \tag{2}$$

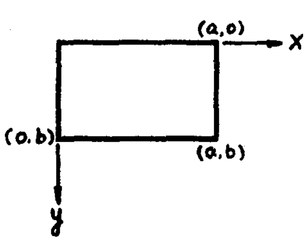


图　4

如果 (x_1, y_1), (x_2, y_2), (x_3, y_3) 三点为矩形壳体之 $(0, b)$, (a, b), $(a, 0)$ 三点,则

$$\left.\begin{aligned} v_1 - v_2 &= -\int_0^a \left(\frac{\partial v}{\partial x}\right)^{y=b} dx, \\ u_3 - u_2 &= -\int_0^b \left(\frac{\partial u}{\partial y}\right)^{x=a} dy. \end{aligned}\right\} \tag{3}$$

将式(3)代入式(2)得

$$\theta' - \theta = -\frac{1}{a}\int_0^a \left(\frac{\partial v}{\partial x}\right)^{y=b} dx - \frac{1}{b}\int_0^b \left(\frac{\partial u}{\partial y}\right)^{x=a} dy. \tag{4}$$

特别当 $\theta' = \theta$ 时

$$a\int_0^b\left(\frac{\partial u}{\partial y}\right)^{x=a}dy+b\int_0^a\left(\frac{\partial v}{\partial x}\right)^{y=b}dx=0. \tag{5}$$

由熟知的壳体应力与位移关系不难得

$$\int_0^b\left(\frac{\partial u}{\partial y}\right)^{x=a}dy=\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b+\int_0^b 2k_{xy}w\,dy+\int_0^b y\left[\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial x}(k_y w)\right]dy-\frac{1+\nu}{Eh}b\frac{\partial^2\varphi}{\partial x\partial y}\Big|^b-b\left(\frac{\partial v}{\partial x}\right)^{y=b}\right\}^{x=a}, \tag{6}$$

$$\int_0^a\left(\frac{\partial v}{\partial x}\right)^{y=b}dx=\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big|_0^a+\int_0^a 2k_{xy}w\,dx+\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial y}(k_x w)\right]dx-\frac{1+\nu}{Eh}a\frac{\partial^2\varphi}{\partial x\partial y}\Big|^a-a\left(\frac{\partial u}{\partial y}\right)^{x=a}\right\}^{y=b}, \tag{7}$$

$$a\int_0^b\left(\frac{\partial u}{\partial y}\right)^{x=a}dy+b\int_0^a\left(\frac{\partial v}{\partial x}\right)^{y=b}dx=\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b\cdot a+a\int_0^b 2k_{xy}w\,dy+\right.$$
$$\left.+a\int_0^b y\left[\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{\partial}{\partial x}(k_y w)\right]dy\right\}^{x=a}+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big|_0^a b+\right.$$
$$\left.+b\int_0^a 2k_{xy}w\,dx+b\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{\partial}{\partial y}(k_x w)\right]dx\right\}^{y=b}-$$
$$-2abk_{xy}w\big|^{(a,b)}. \tag{8}$$

例 4　矩形底三边简支一边自由.

由前知需两个补充条件. 因壳体外部多余两个连杆，故补充条件表示位移的特征，即 $\angle ABC$ 保持直角及点 A, B 无相对位移. 由公式(8)，将 $x=a$ 及 $y=b$ 的简支条件 $w=0$ 及自由边 $\frac{\partial^2\varphi}{\partial x\partial y}=0$ 代入，得

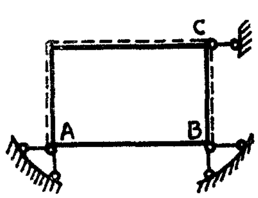


图　5

$$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b a+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right)dy\right\}^{x=a}+$$
$$+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big|_0^a b+b\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial y}(k_x w)\right]dx+2b\int_0^a k_{xy}w\,dx\right\}^{y=b}=0. \tag{9}$$

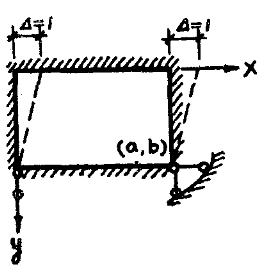


图　6

又 $y=b$ 之点 A, B 无相对位移，即

$$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0. \tag{10}$$

式(9)和(10)即所求之补充条件.

例 5　四边固定，有三根外连杆.

根据公式(1)得 $n=1$，即只需一个补充条件，又由 $n_1=1$ 知补充条件系表示力的特征，且可用虚位移原理求出：当壳体产生一可能位移 $\Delta=1$ 时，

$$\int_0^b\left[\left(N_x\frac{b-y}{b}\right)^{x=a}-\left(N_x\frac{b-y}{b}\right)^{x=0}\right]dy-\int_0^a(N_{xy})^{y=0}dx=0.$$

将应力表为应力函数，经分部积分后得

$$\varphi\big|^{(a,b)}+\varphi\big|^{(0,0)}-\varphi\big|^{(a,0)}-\varphi\big|^{(0,b)}=0.$$

此即所求之补充条件.

例 6　四边简支，考虑柱子的水平刚度，支承情况见图 7. 点 $(a,\ 0)$ 为一弹簧，弹簧系数为 K，点 $(a,\ 0)$ 的位移为 c_1. 由式(4)得

$$\frac{c_1}{b}=-\frac{1}{a}\int_0^a\left(\frac{\partial v}{\partial x}\right)^{y=b}dx-\frac{1}{b}\int_0^b\left(\frac{\partial u}{\partial y}\right)^{x=a}dy,\qquad(11)$$

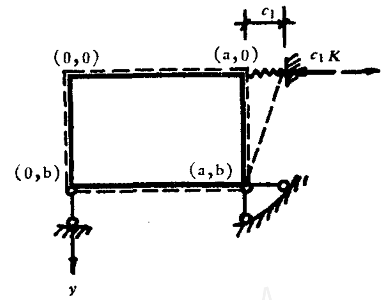


图　7

又由虚位移原理得

$$-\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=c_1K.\qquad(12)$$

将式(11)与(12)代入式(8)，即得补充条件

$$\frac{a}{K}\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b a+\right.$$
$$\left.+a\int_0^b y\left[\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right]\right\}^{x=a}+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big|_0^a\cdot b+\right.$$
$$\left.+b\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+k_x\frac{\partial w}{\partial y}\right]dx\right\}^{y=b}.\qquad(13)$$

例 7　七边简支，有七根外连杆.

$n=4$，补充条件为(其意义为图中每虚线两端无相对位移)

$$\int_{C_{ij}}\varepsilon_s ds=0,\qquad(14)$$

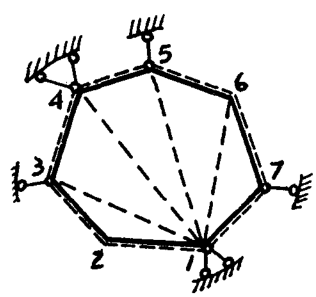


图　8

其中线积分路线 C_{ij} 分别为 C_{13}, C_{14}, C_{15}, C_{16}, 即各 i, j 两点之连线，s 为沿各 C_{ij} 之长度坐标，又

$$\varepsilon_s=\frac{1}{Eh}(N_s-\nu N_n)+k_s w,$$
$$N_s=N_x m^2-2N_{xy}lm+N_y l^2,$$
$$N_n=N_x l^2+2N_{xy}lm+N_y m^2,$$
$$k_s=k_x l^2+2k_{xy}lm+k_y m^2,$$
$$l=\cos(n,\ x),$$
$$m=\cos(n,\ y).$$

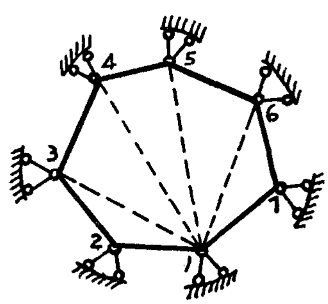


图　9

例 8　七边自由，每角点有两根外连杆.

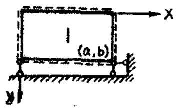
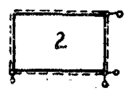
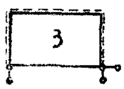
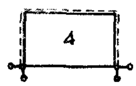
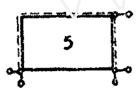
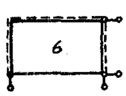
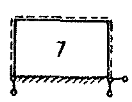
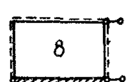
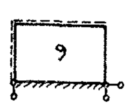
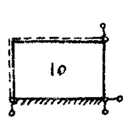
$n=11$，补充条件为

$$\int_{C_{ij}}\varepsilon_s ds=0,$$

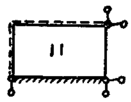
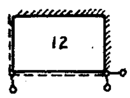
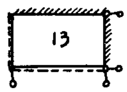
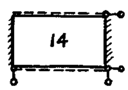
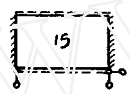
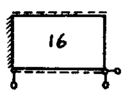
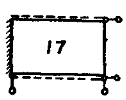
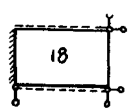
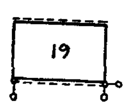
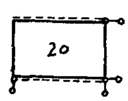
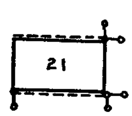
其中 C_{ij} 分别为 C_{12}, C_{23}, C_{34}, C_{45}, C_{56}, C_{67}, C_{71}, C_{13}, C_{14}, C_{15}, C_{16}, 意义同前例.

推导矩形底的补充条件有时亦可用表示壳体内部的应变积分式如式(14)，但矩形底扁壳用直角坐标更便于计算.

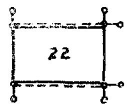
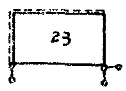
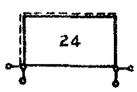
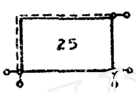
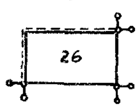
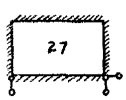
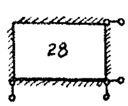
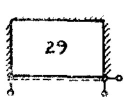
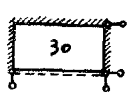
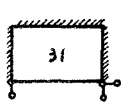
附表　矩形底扁壳边界补充条件

	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\vert_0^b a+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right)dy\right\}^{x=a}+$ $+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\vert_0^a b+b\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+k_x\frac{\partial w}{\partial y}\right)dx\right\}^{y=b}=0$ [注：上式与 $A=2Gh\int_0^a\int_0^b wk_{xy}dxdy$（$A$ 为某一角点之 φ，其他三角点 $\varphi=0$）一致.]
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$ $\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\vert_0^b\cdot a+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right)dy\right\}^{x=a}+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\vert_0^a b+\right.$ $\left.+b\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{\partial k_xw}{\partial y}\right)dx+2b\int_0^a k_{xy}wdx\right\}^{y=b}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\vert_0^b a+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right)dy\right\}^{x=a}+$ $+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\vert_0^a b+b\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{\partial k_xw}{\partial y}\right)dx+\right.$ $\left.+2b\int_0^a k_{xy}wdx\right\}^{y=b}=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+k_y\frac{\partial w}{\partial x}\right)dy-\frac{1+\nu}{Eh}\left(b\frac{\partial^2\varphi}{\partial x\partial y}\Big\vert^b+\frac{\partial\varphi}{\partial x}\Big\vert_0^b\right)=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_yw\right)^{x=a}dy=0$

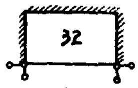
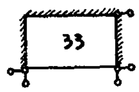
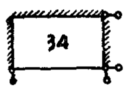
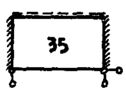
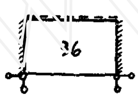
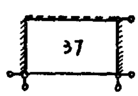
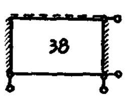
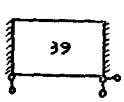
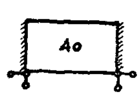
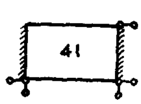
续　上　表

	$\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=a}dy=0$ $\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\|_0^b+\int_0^b 2k_{xy}w\,dy+\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{\partial k_y w}{\partial x}\right)dy\right\}^{x=a}=0$
	$\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\frac{1}{b}\varphi\Big\|_{(a,0)}^{(a,b)}=0$
	角点 $(a,0)$，$\frac{\partial^2\varphi}{\partial x\partial y}=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\|_0^a+\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+k_x\frac{\partial w}{\partial y}\right)dx-\right.$ $\left.-\frac{1+\nu}{Eh}\frac{\partial^2\varphi}{\partial x\partial y}\Big\|^a a\right\}^{y=b}=0$
	$\varphi\|^{(0,0)}+\varphi\|^{(a,b)}-\varphi\|^{(a,0)}-\varphi\|^{(0,b)}=0$
	$\frac{\partial\varphi}{\partial y}\Big\|^{(a,0)}-\frac{1}{b}\varphi\Big\|_{(a,0)}^{(a,b)}=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\|_0^b a+a\int_0^b 2k_{xy}w\,dy+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{\partial k_y w}{\partial x}\right)dy\right\}^{x=a}+$ $+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\|_0^a b+b\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+k_x\frac{\partial w}{\partial y}\right)dx\right\}^{y=b}=0$
	同情形 17 之第一条件 $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=a}dy=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	同情形 17 之第一条件
	同情形 17 之第一条件 $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=a}dy=0$

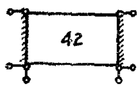
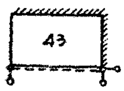
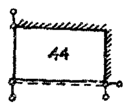
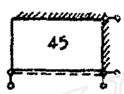
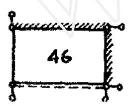
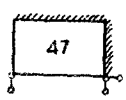
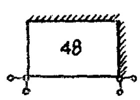
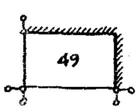
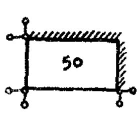
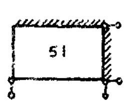
续 上 表

	同情形 17 之第一条件 $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=0}dy=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=a}dy=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$
	$\int_0^a\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^{y=0}dx=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$
	$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big\|_0^b a+a\int_0^b 2k_{xy}w\,dy+a\int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{\partial k_y w}{\partial x}\right)dy\right\}^{x=a}+$ $+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\|_0^a b+b\int_0^a 2k_{xy}w\,dx+b\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{\partial k_x w}{\partial y}\right)dx\right\}^{y=b}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$
	同情形 25 之第一条件 $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=a}dy=0$
	$\varphi\|^{(0,0)}+\varphi\|^{(a,b)}-\varphi\|^{(0,b)}-\varphi\|^{(a,0)}=0$
	四角点任取一点 $\frac{\partial^2\varphi}{\partial x\partial y}=0$
	$\varphi\|^{(0,0)}+\varphi\|^{(a,b)}-\varphi\|^{(0,b)}-\varphi\|^{(a,0)}=0$
	角点(0,0)或(a,0)，$\frac{\partial^2\varphi}{\partial x\partial y}=0$
	$\varphi\|^{(0,0)}+\varphi\|^{(a,b)}-\varphi\|^{(0,b)}-\varphi\|^{(a,0)}=0$ $b\frac{\partial\varphi}{\partial y}\Big\|^{(0,b)}-\varphi\Big\|_{(0,0)}^{(0,b)}=0$

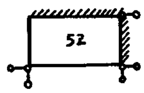
续　上　表

 32	$\varphi\big\|^{(0,0)}+\varphi\big\|^{(a,b)}-\varphi\big\|^{(0,b)}-\varphi\big\|^{(a,0)}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
 33	角点 (0,0) 或 (a,0), $\frac{\partial^2\varphi}{\partial x\partial y}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
 34	$b\frac{\partial\varphi}{\partial y}\Big\|^{(0,b)}-\varphi\Big\|_{(0,0)}^{(0,b)}=0$ 角点 (a,b), $\frac{\partial^2\varphi}{\partial x\partial y}=0$
 35	同情形 31 之第一及第二条件
 36	$\varphi\big\|^{(0,0)}+\varphi\big\|^{(a,b)}-\varphi\big\|^{(0,b)}-\varphi\big\|^{(a,0)}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
 37	$\left\{\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{\partial}{\partial y}(k_xw)\right]dx+\int_0^a 2k_{xy}w\,dx\right\}^{y=b}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
 38	同情形 37 之第一条件 $b\frac{\partial\varphi}{\partial y}\Big\|^{(0,b)}-\varphi\Big\|_{(0,0)}^{(0,b)}=0$
 39	$b\frac{\partial\varphi}{\partial y}\Big\|^{(0,b)}-\varphi\Big\|_{(0,0)}^{(0,b)}=0$ $b\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(0,b)}=0$ $b\frac{\partial\varphi}{\partial y}\Big\|_{(a,0)}+\varphi\Big\|_{(a,0)}^{(a,b)}=0$
 40	$b\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(0,b)}=0$ $b\frac{\partial\varphi}{\partial y}\Big\|_{(a,0)}+\varphi\Big\|_{(a,0)}^{(a,b)}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$
 41	同情形 37 之第一条件 $b\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(0,b)}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_xw\right)^{y=b}dx=0$

续 上 表

	同情形 37 之第一条件 $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=0}dx=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$
	$a\frac{\partial\varphi}{\partial x}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(a,0)}=0$ $b\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\varphi\Big\|_{(a,0)}^{(a,b)}=0$
	$b\frac{\partial\varphi}{\partial y}\Big\|_{(0,0)}+\varphi\Big\|_{(a,0)}^{(a,b)}=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=0}dy=0$
	$a\frac{\partial\varphi}{\partial x}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(a,0)}=0$ $\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big\|_0^a+\int_0^a x\left(\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi+k_x\frac{\partial w}{\partial y}\right)dx-\frac{1+\nu}{Eh}\frac{\partial^2\varphi}{\partial x\partial y}\Big\|^a a\right\}^{y=b}=0$
	角点 $(a, 0)$，$\frac{\partial^2\varphi}{\partial x\partial y}=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=0}dy=0$
	同情形 43 之第一、二条件
	同情形 43 之第一、二条件 $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$
	同情形 44 之第一条件 $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=0}dy=0$
	角点 $(a, 0)$，$\frac{\partial^2\varphi}{\partial x\partial y}=0$ $\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2}+k_x w\right)^{y=b}dx=0$ $\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2}+k_y w\right)^{x=0}dy=0$
	同情形 37 之第一条件 $a\frac{\partial\varphi}{\partial x}\Big\|_{(0,0)}+\varphi\Big\|_{(0,0)}^{(a,0)}=0$

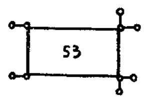
续　上　表



$$a\frac{\partial\varphi}{\partial x}\Big|_{(0,0)} + \varphi\Big|_{(0,0)}^{(a,0)} = 0$$

$$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2} + k_x w\right)^{y=b} dx = 0$$

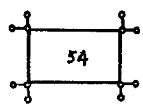
同情形 37 之第一条件



$$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2} + k_x w\right)^{y=0} dx = 0$$

$$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2} + k_x w\right)^{y=b} dx = 0$$

$$\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2} + k_y w\right)^{x=a} dy = 0$$

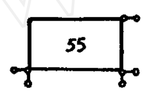


同情形 53 之第一、二、三条件

$$\int_0^b\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial x^2} + k_y w\right)^{x=0} dy = 0$$

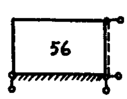
$$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b a + a\int_0^b 2k_{xy} w\,dy + a\int_0^b y\left[\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi + \frac{\partial}{\partial x}(k_y w)\right]dy\right\}^{x=a} +$$

$$+\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial y}\Big|_0^a b + b\int_0^a 2k_{xy} w\,dx + b\int_0^a x\left[\frac{1}{Eh}\frac{\partial}{\partial y}\nabla^2\varphi + \frac{\partial}{\partial y}(k_x w)\right]dx\right\}^{y=b} = 0$$



$$\int_0^a\left(\frac{1}{Eh}\frac{\partial^2\varphi}{\partial y^2} + k_x w\right)^{y=b} dx = 0$$

同情形 54 之第五条件



$$\left\{-\frac{1+\nu}{Eh}\frac{\partial\varphi}{\partial x}\Big|_0^b + \int_0^b y\left(\frac{1}{Eh}\frac{\partial}{\partial x}\nabla^2\varphi + k_y\frac{\partial w}{\partial x}\right)dy - \right.$$

$$\left. -\frac{1+\nu}{Eh}b\frac{\partial^2\varphi}{\partial x\partial y}\Big|^b\right\}^{x=a} = 0$$

說明:

1. 图例: ////////// 固定边, ═══════ 简支边, ———— 自由边.
2. 所有坐标及壳底边长均同情况 1.
3. 所有连杆均与同一刚体相连如情形 1.
4. 角点 $w \neq 0$ 的情形可按本文方法推导.

参　考　文　献

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ON THE SUPPLEMENTARY CONDITION TO THE BOUNDARY PROBLEM OF ELASTIC SHALLOW SHELLS

PEI WEN-JIN

(*The Northwest Design Institute of Industrial Architecture, Ministry of Building*)