第 8 卷 第 4 期　　力 学 学 报　　Vol. 8, No. 4
1965 年 12 月　　ACTA MECHANICA SINICA　　Dec., 1965

矩形截面异质柱体在半无限弹性体內的热应力問題*

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一、含有矩形截面异质柱的无限弹性体的热应力問題

如图 1 所示，无限弹性体內装有矩形截面异质柱，柱轴平行于 z 轴，设 $c > 0$，异质柱的热胀系数 α_i 大于无限弹性体的热胀系数 α_a，但两种材料有相同的弹性系数．当温度等于零时，这个组合体里无应力存在；当組合体的温度 T_0 大于零时，试求解这个組合体的热应力問題．

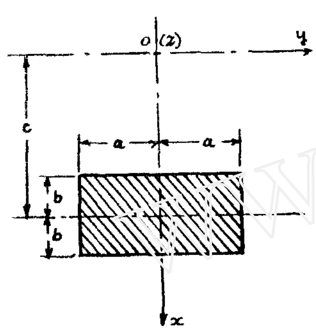

图 1　含有矩形截面异质体的无限弹性体

按热弹性理论，这里的有关微分方程是

$$\nabla^2\Phi = \frac{1+\nu}{1-\nu}\alpha T = \left.\begin{cases} K, \text{当 } c-b \leqslant x \leqslant c+b,\ -a < y < a \text{ 时}, \\ 0, \text{当 } x < c-b,\ x > c+b,\ y < -a,\ y > a \text{ 时}, \end{cases}\right\} \quad (1)$$

其中 $K = \dfrac{1+\nu}{1-\nu}\eta_0 T_0$，$\eta_0 = \alpha_i - \alpha_a$，$\nu$ 是 Poisson 比，$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$．

按照场论，公式(1)的解答是热弹性位移势函数 Φ：

$$\Phi = \frac{K}{2\pi}\int_{\xi=c-b}^{c+b}\int_{\eta=-a}^{a} \ln r\, d\xi d\eta, \qquad r = \sqrt{(x-\xi)^2 + (y-\eta)^2}. \quad (2)$$

以 G 代表剪切模量，相应于 Φ 的应力和位移是：

$$\begin{aligned} \bar{\sigma}_x &= -2G\frac{\partial^2\Phi}{\partial y^2} = -\frac{GK}{\pi}\iint_A \frac{(x-\xi)^2 - (y-\eta)^2}{[(x-\xi)^2 + (y-\eta)^2]^2} d\xi d\eta \\ &= -\frac{GK}{\pi}\int_{x-c+b}^{x-c-b}\left[\frac{y-a}{\xi^2 + (y-a)^2} - \frac{y+a}{\xi^2 + (y+a)^2}\right] d\xi \\ &= -\frac{GK}{\pi}\left[\operatorname{arc\,tg}\frac{x-c-b}{y-a} - \operatorname{arc\,tg}\frac{x-c+b}{y-a} - \right. \\ &\quad \left. - \operatorname{arc\,tg}\frac{x-c-b}{y+a} + \operatorname{arc\,tg}\frac{x-c+b}{y+a}\right], \end{aligned} \quad (3)$$

$$\bar{\sigma}_y = -2G\frac{\partial^2\Phi}{\partial x^2} = -\frac{GK}{\pi}\iint_A \frac{(y-\eta)^2 - (x-\xi)^2}{[(x-\xi)^2 + (y-\eta)^2]^2} d\xi d\eta$$

* 1964 年 7 月 13 日收到．

$$= -\frac{GK}{\pi}\left[\operatorname{arc\,tg}\frac{y-a}{x-c-b} - \operatorname{arc\,tg}\frac{y+a}{x-c-b} - \right.$$

$$\left. - \operatorname{arc\,tg}\frac{y-a}{x-c+b} + \operatorname{arc\,tg}\frac{y+a}{x-c+b}\right], \tag{4}$$

$$\bar{\tau}_{xy} = 2G\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{2GK}{\pi}\iint_A \frac{(x-\xi)(y-\eta)}{[(x-\xi)^2+(y-\eta)^2]^2}d\xi d\eta$$

$$= -\frac{GK}{\pi}\int_{c-b}^{c+b}\left[\frac{x-\xi}{(x-\xi)^2+(y-a)^2} - \frac{x-\xi}{(x-\xi)^2+(y+a)^2}\right]d\xi$$

$$= \frac{GK}{2\pi}\ln\frac{[(x-c-b)^2+(y-a)^2][(x-c+b)^2+(y+a)^2]}{[(x-c-b)^2+(y+a)^2][(x-c+b)^2+(y-a)^2]}, \tag{5}$$

利用关系 $\varphi \gtrless 0$ 时 $\operatorname{arc\,tg}\varphi + \operatorname{arc\,tg}\frac{1}{\varphi} = \pm\frac{\pi}{2}$,就可验证

$$\bar{\sigma}_z = -2G\nabla^2\Phi = \begin{cases} 2GK, & \text{在矩形截面之内,} \\ 0, & \text{在矩形截面之外;} \end{cases} \tag{6}$$

$$\bar{v} = \frac{\partial\Phi}{\partial y} = \frac{K}{2\pi}\iint_A \frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}d\xi d\eta = \frac{K}{4\pi}\int_{x-c+b}^{x-c-b}\ln\frac{\xi^2+(y-a)^2}{\xi^2+(y+a)^2}d\xi =$$

$$= \frac{K}{4\pi}\left\{(x-c-b)\ln\frac{(x-c-b)^2+(y-a)^2}{(x-c-b)^2+(y+a)^2} - \right.$$

$$-(x-c+b)\ln\frac{(x-c+b)^2+(y-a)^2}{(x-c+b)^2+(y+a)^2} +$$

$$+2(y-a)\left[\operatorname{arc\,tg}\frac{x-c-b}{y-a} - \operatorname{arc\,tg}\frac{x-c+b}{y-a}\right] -$$

$$\left. -2(y+a)\left[\operatorname{arc\,tg}\frac{x-c-b}{y+a} - \operatorname{arc\,tg}\frac{x-c+b}{y+a}\right]\right\}, \tag{7}$$

$$\bar{u} = \frac{\partial\Phi}{\partial x} = \frac{K}{2\pi}\iint_A \frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}d\xi d\eta = \frac{K}{4\pi}\int_{y+a}^{y-a}\ln\frac{(x-c-b)^2+\bar{\eta}^2}{(x-c+b)^2+\bar{\eta}^2}d\bar{\eta} =$$

$$= \frac{K}{4\pi}\left\{(y-a)\ln\frac{(x-c-b)^2+(y-a)^2}{(x-c+b)^2+(y-a)^2} - \right.$$

$$-(y+a)\ln\frac{(x-c-b)^2+(y+a)^2}{(x-c+b)^2+(y+a)^2} +$$

$$+2(x-c-b)\left[\operatorname{arc\,tg}\frac{y-a}{x-c-b} - \operatorname{arc\,tg}\frac{y+a}{x-c-b}\right] -$$

$$\left. -2(x-c+b)\left[\operatorname{arc\,tg}\frac{y-a}{x-c+b} - \operatorname{arc\,tg}\frac{y+a}{x-c+b}\right]\right\}, \tag{8}$$

其中双积分号下的 A 代表矩形截面的面积．公式(3)至(8)即本节问题的解．当然，若仅求解本节的问题，应使 $c = 0$ 以求简化运算．

二、半无限弹性組合体的弹性力学边界值問題

现在考虑矩形截面异质体和半无限弹性体所组成的组合体 $x \geqq 0$，$-\infty < z < \infty$ 和 $c > b$ 的热应力问题．由于

$$(\bar{\sigma}_x)_{x=0}=\frac{GK}{\pi}\left[\operatorname{arc\,tg}\frac{c+b}{y-a}-\operatorname{arc\,tg}\frac{c-b}{y-a}-\operatorname{arc\,tg}\frac{c+b}{y+a}+\operatorname{arc\,tg}\frac{c-b}{y+a}\right]\neq 0$$

和

$$(\bar{\tau}_{xy})_{x=0}=\frac{GK}{2\pi}\ln\frac{[(c+b)^2+(y-a)^2][(c-b)^2+(y+a)^2]}{[(c+b)^2+(y+a)^2][(c-b)^2+(y-a)^2]}\neq 0,$$

所以须再解决保证半无限弹性组合体须具有自由表面的弹性力学边界值问题. 为此, 先利用分部积分法由

$$\bar{\bar{\sigma}}_0=-\int_{-\infty}^{\infty}(\bar{\sigma}_x)_{x=0}e^{i\xi y}dy=-\frac{2GK}{\pi}\int_0^{\infty}\left[\operatorname{arc\,tg}\frac{c+b}{y-a}-\right.$$

$$\left.-\operatorname{arc\,tg}\frac{c-b}{y-a}-\operatorname{arc\,tg}\frac{c+b}{y+a}+\operatorname{arc\,tg}\frac{c-b}{y+a}\right]\cos(\xi y)dy$$

导得

$$\bar{\bar{\sigma}}_0=-\frac{2GK}{\pi}\int_0^{\infty}\frac{\sin(\xi y)}{\xi}\left[\frac{c+b}{(c+b)^2+(y-a)^2}-\frac{c-b}{(c-b)^2+(y-a)^2}-\right.$$

$$\left.-\frac{c+b}{(c+b)^2+(y+a)^2}+\frac{c-b}{(c-b)^2+(y+a)^2}\right]dy,$$

再利用变换 $y-a=y_1$ 和 $y+a=y_2$, 把上式化成

$$\bar{\bar{\sigma}}_0=-\frac{2GK}{\pi\xi}\left(\int_{-a}^{0}+\int_0^{\infty}\right)(\cos(\xi a)\sin(\xi y_1)+\sin(\xi a)\cos(\xi y_1))\times$$

$$\times\left[\frac{c+b}{(c+b)^2+y_1^2}-\frac{c-b}{(c-b)^2+y_1^2}\right]dy_1+$$

$$+\frac{2GK}{\pi\xi}\left(\int_{+a}^{0}+\int_0^{\infty}\right)(\cos(\xi a)\sin(\xi y_2)-\sin(\xi a)\cos(\xi y_2))\times$$

$$\times\left[\frac{c+b}{(c+b)^2+y_2^2}-\frac{c-b}{(c-b)^2+y_2^2}\right]dy_2=$$

$$=-\frac{2GK}{\pi}\frac{\sin(\xi a)}{\xi}\int_0^{\infty}\left\{(c+b)\left[\frac{\cos(\xi y_1)}{(c+b)^2+y_1^2}dy_1+\frac{\cos(\xi y_2)}{(c+b)^2+y_2^2}dy_2\right]-\right.$$

$$\left.-(c-b)\left[\frac{\cos(\xi y_1)}{(c-b)^2+y_1^2}dy_1+\frac{\cos(\xi y_2)}{(c-b)^2+y_2^2}dy_2\right]\right\},$$

最后利用 $\int_0^{\infty}\frac{\cos(\alpha x)}{\alpha^2+k^2}d\alpha=\frac{\pi}{2k}e^{-kx}$, 其中 $k>0$, 即得

$$\bar{\bar{\sigma}}_0=-2GK\frac{\sin(\xi a)}{\xi}(e^{-(c+b)|\xi|}-e^{-(c-b)|\xi|}). \tag{9}$$

另一方面,可利用变换 $y-a=y_1$ 和 $y+a=y_2$ 把

$$\bar{\bar{\tau}}_0=-\int_{-\infty}^{\infty}(\bar{\tau}_{xy})_{x=0}e^{i\xi y}dy=$$

$$=-\frac{GKi}{\pi}\int_0^{\infty}\ln\frac{[(c+b)^2+(y-a)^2][(c-b)^2+(y+a)^2]}{[(c+b)^2+(y+a)^2][(c-b)^2+(y-a)^2]}\sin(\xi y)dy$$

化为

$$\bar{\bar{\tau}}_0=-\frac{GKi}{\pi}\left\{\left(\int_{-a}^{0}+\int_0^{\infty}\right)[\cos(\xi a)\sin(\xi y_1)+\sin(\xi a)\cos(\xi y_1)]\times\right.$$

$$\times \ln \frac{(c+b)^2+y_1^2}{(c-b)^2+y_1^2} dy_1 +$$

$$+\left(\int_{+a}^{0}+\int_{0}^{\infty}\right)[\cos(\xi a)\sin(\xi y_2)-\sin(\xi a)\cos(\xi y_2)]\times$$

$$\left.\times \ln \frac{(c-b)^2+y_2^2}{(c+b)^2+y_2^2} dy_2\right\} =$$

$$= \frac{2GKi}{\pi}\sin(\xi a)\int_0^\infty \ln\frac{(c-b)^2+y_3^2}{(c+b)^2+y_3^2}\cos(\xi y_3)dy_3;$$

又因

$$\pi\frac{e^{-\beta y}-e^{-\alpha y}}{y}=\int_0^\infty \ln\frac{\alpha^2+x^2}{\beta^2+x^2}\cos(xy)dx,\ y>0,\ \mathrm{Re}\,\alpha>0,\ \mathrm{Re}\,\beta>0,$$

所以导得

$$\bar{\bar{\tau}}_0 = 2GKi\frac{\sin(\xi a)}{|\xi|}\left(e^{-(c+b)|\xi|}-e^{-(c-b)|\xi|}\right). \tag{10}$$

现在若把公式(9)和(10)代入参考 Fourier 变换[1]而导出的下列诸式：

$$\left.\begin{aligned}
\bar{\bar{\sigma}}_x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\{[1+|\xi|x]\bar{\bar{\sigma}}_0+ix\xi\bar{\bar{\tau}}_0\}d\xi,\\
\bar{\bar{\sigma}}_y &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\left\{[1-|\xi|x]\bar{\bar{\sigma}}_0+i\frac{2|\xi|-\xi^2x}{\xi}\bar{\bar{\tau}}_0\right\}d\xi,\\
\bar{\bar{\tau}}_{xy} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\{ix\xi\bar{\bar{\sigma}}_0+[1-|\xi|x]\bar{\bar{\tau}}_0\}d\xi,\\
\bar{\bar{\sigma}}_z &= \frac{\nu}{\pi}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\left[\bar{\bar{\sigma}}_0+i\frac{|\xi|}{\xi}\bar{\bar{\tau}}_0\right]d\xi,\\
\bar{\bar{v}} &= \frac{1}{4\pi G}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\left\{[1-2\nu-|\xi|x]\frac{i}{\xi}\bar{\bar{\sigma}}_0-\left[\frac{2(1-\nu)}{|\xi|}-x\right]\bar{\bar{\tau}}_0\right\}d\xi,\\
\bar{\bar{u}} &= -\frac{1}{4\pi G}\int_{-\infty}^{\infty} e^{-|\xi|x-i\xi y}\left\{\left[\frac{2(1-\nu)}{|\xi|}+x\right]\bar{\bar{\sigma}}_0+[1-2\nu+|\xi|x]\frac{i}{\xi}\bar{\bar{\tau}}_0\right\}d\xi,
\end{aligned}\right\} \tag{11}$$

并令 $\beta_1=c+b+x$ 和 $\beta_2=c-b+x$，则得

$$\left.\begin{aligned}
\bar{\bar{\sigma}}_x &= -\frac{2GK}{\pi}(J_1+2xJ_2), \quad \bar{\bar{\sigma}}_y = -\frac{2GK}{\pi}(3J_1-2xJ_2),\\
\bar{\bar{\tau}}_{xy} &= -\frac{2GK}{\pi}(2xJ_3-J_4), \quad \bar{\bar{\sigma}}_z = -\frac{8\nu GK}{\pi}J_1,\\
\bar{\bar{v}} &= -\frac{K}{\pi}[(3-4\nu)J_5-2xJ_4], \quad \bar{\bar{u}} = \frac{K}{\pi}[(3-4\nu)J_6+2xJ_1],
\end{aligned}\right\} \tag{12}$$

其中

$$\left.\begin{aligned}
J_1 &= \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi}-e^{-\beta_2\xi})\frac{\sin[\xi(a+y)]+\sin[\xi(a-y)]}{\xi}d\xi,\\
J_2 &= \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi}-e^{-\beta_2\xi})\{\sin[\xi(a+y)]+\sin[\xi(a-y)]\}d\xi,\\
J_3 &= \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi}-e^{-\beta_2\xi})\{\cos[\xi(a-y)]-\cos[\xi(a+y)]\}d\xi,
\end{aligned}\right\} \tag{13}$$

$$J_4 = \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi} - e^{-\beta_2\xi}) \frac{\cos[\xi(a-y)] - \cos[\xi(a+y)]}{\xi} d\xi,$$

$$J_5 = \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi} - e^{-\beta_2\xi}) \frac{\cos[\xi(a-y)] - \cos[\xi(a+y)]}{\xi^2} d\xi,$$

$$J_6 = \frac{1}{2}\int_0^\infty (e^{-\beta_1\xi} - e^{-\beta_2\xi}) \frac{\sin[\xi(a+y)] + \sin[\xi(a-y)]}{\xi^2} d\xi.$$

利用下列诸式[2]:

$$\int_0^\infty \frac{e^{-\lambda\alpha}}{\alpha}\sin(\alpha\mu)d\alpha = \text{arc tg}\,\frac{\mu}{\lambda}, \quad \int_0^\infty e^{-\lambda\alpha}\sin(\alpha\mu)d\alpha = \frac{\mu}{\lambda^2+\mu^2},$$

$$\int_0^\infty e^{-\lambda\alpha}\cos(\alpha\mu)d\alpha = \frac{\lambda}{\lambda^2+\mu^2}, \quad \int_0^\infty \frac{e^{-\lambda_0\alpha} - e^{-\lambda_1\alpha}}{\alpha}\cos(\alpha\mu)d\alpha = \frac{1}{2}\ln\frac{\lambda_1^2+\mu^2}{\lambda_0^2+\mu^2}$$

$$(\lambda > 0,\ \lambda_0 > 0,\ \lambda_1 > 0),$$

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x^2}\sin(x\lambda)dx = \frac{1}{2}\lambda\ln\frac{\lambda^2+\beta^2}{\lambda^2+\alpha^2} + \beta\,\text{arc tg}\,\frac{\lambda}{\beta} - \alpha\,\text{arc tg}\,\frac{\lambda}{\alpha}$$

$$(\lambda > 0,\ \text{Re}\,\alpha > 0,\ \text{Re}\,\beta > 0),$$

就可算出 J_1 至 J_4 和 J_6 的积分:

$$\begin{aligned}
J_1 &= \frac{1}{2}\Big[\text{arc tg}\,\frac{a+y}{c+b+x} - \text{arc tg}\,\frac{a+y}{c-b+x} + \\
&\quad + \text{arc tg}\,\frac{a-y}{c+b+x} - \text{arc tg}\,\frac{a-y}{c-b+x}\Big], \\
J_2 &= \frac{1}{2}\Big[\frac{a+y}{(a+y)^2+(c+b+x)^2} - \frac{a+y}{(a+y)^2+(c-b+x)^2} + \\
&\quad + \frac{a-y}{(a-y)^2+(c+b+x)^2} - \frac{a-y}{(a-y)^2+(c-b+x)^2}\Big], \\
J_3 &= \frac{1}{2}\Big[\frac{c+b+x}{(a-y)^2+(c+b+x)^2} - \frac{c-b+x}{(a-y)^2+(c-b+x)^2} - \\
&\quad - \frac{c+b+x}{(a+y)^2+(c+b+x)^2} + \frac{c-b+x}{(a+y)^2+(c-b+x)^2}\Big], \\
J_4 &= \frac{1}{4}\ln\frac{[(c-b+x)^2+(a-y)^2][(c+b+x)^2+(a+y)^2]}{[(c+b+x)^2+(a-y)^2][(c-b+x)^2+(a+y)^2]}, \\
J_6 &= \frac{1}{4}\Big\{(a+y)\ln\frac{(a+y)^2+(c-b+x)^2}{(a+y)^2+(c+b+x)^2} + \\
&\quad + (a-y)\ln\frac{(a-y)^2+(c-b+x)^2}{(a-y)^2+(c+b+x)^2}\Big\} + \\
&\quad + \frac{c-b+x}{2}\Big[\text{arc tg}\,\frac{a+y}{c-b+x} + \text{arc tg}\,\frac{a-y}{c-b+x}\Big] - \\
&\quad - \frac{c+b+x}{2}\Big[\text{arc tg}\,\frac{a+y}{c+b+x} + \text{arc tg}\,\frac{a-y}{c+b+x}\Big].
\end{aligned} \tag{14}$$

为要求积 J_5, 引用 $\cos[\xi|a\mp y|] = \frac{1}{2}(e^{i\xi|a\mp y|} + e^{-i\xi|a\mp y|})$ 先把 J_5 写成

$$J_5 = \frac{1}{4}(I_1 + I_2 + I_3 + I_4),$$

其中

$$I_{1,2}=\int_0^\infty\left[e^{-\beta_1\xi\pm i\xi|a-y|}-e^{-\beta_2\xi\pm i\xi|a-y|}\right]\frac{d\xi}{\xi^2},$$

$$I_{3,4}=-\int_0^\infty\left[e^{-\beta_1\xi\pm i\xi|a+y|}-e^{-\beta_2\xi\pm i\xi|a+y|}\right]\frac{d\xi}{\xi^2}.$$

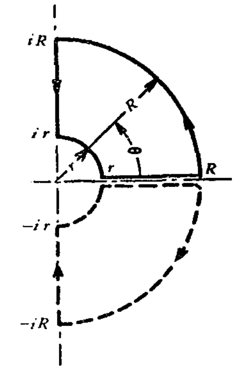

图 2　ξ 复平面内的两条积分路线

设想 $I_{1,2}$ 和 $I_{3,4}$ 均有 Cauchy 主値，依图 2 里的合闭实线并借助于 $\xi=Re^{i\theta}$ 和 $\xi=re^{i\theta}$ 把 I_1 写成

$$\int_r^R\left[e^{-\beta_1\xi+i\xi|a-y|}-e^{-\beta_2\xi+i\xi|a-y|}\right]\frac{d\xi}{\xi^2}+$$

$$+\int_0^{\frac{\pi}{2}}\left[e^{-[\beta_1-i|a-y|]Re^{i\theta}}-e^{-[\beta_2-i|a-y|]Re^{i\theta}}\right]\frac{id\theta}{Re^{i\theta}}+$$

$$+\int_{iR}^{ir}\left[e^{-\beta_1\xi+i\xi|a-y|}-e^{-\beta_2\xi+i\xi|a-y|}\right]\frac{d\xi}{\xi^2}+$$

$$+\int_{\frac{\pi}{2}}^{0}\left[e^{-[\beta_1-i|a-y|]re^{i\theta}}-e^{-[\beta_2-i|a-y|]re^{i\theta}}\right]\frac{id\theta}{re^{i\theta}}=0. \tag{15}$$

使 $R\to\infty$ 和 $r\to 0$，并因 $\cos\theta>0$, $\sin\theta>0$, $\beta_1\cos\theta+|a-y|\sin\theta>0$ 和 $\beta_2\cos\theta+|a-y|\sin\theta>0$，所以自公式(15)和 I_3 的相应的式子导得

$$I_{1,3}=\mp i\int_0^\infty\left\{e^{-[\beta_1-i|a\mp y|]i\xi}-e^{-[\beta_2-i|a\mp y|]i\xi}\right\}\frac{d\xi}{\xi^2}\pm i(\beta_2-\beta_1)\frac{\pi}{2}; \tag{16}$$

依同理可把 I_2 依图 2 的合闭虚线写成

$$\int_r^R\left\{e^{-[\beta_1+i|a-y|]\xi}-e^{-[\beta_2+i|a-y|]\xi}\right\}\frac{d\xi}{\xi^2}+$$

$$+\int_0^{-\frac{\pi}{2}}\left\{e^{-[\beta_1+i|a-y|]Re^{i\theta}}-e^{-[\beta_2+i|a-y|]Re^{i\theta}}\right\}\frac{id\theta}{Re^{i\theta}}+$$

$$+\int_{-iR}^{-ir}\left\{e^{-[\beta_1+i|a-y|]\xi}-e^{-[\beta_2+i|a-y|]\xi}\right\}\frac{d\xi}{\xi^2}+$$

$$+\int_{-\frac{\pi}{2}}^{0}\left\{e^{-[\beta_1+i|a-y|]re^{i\theta}}-e^{-[\beta_1+i|a-y|]re^{i\theta}}\right\}\frac{id\theta}{re^{i\theta}}=0. \tag{17}$$

因为 $\sin\theta<0$, $\cos\theta>0$, $\beta_1\cos\theta-|a-y|\sin\theta>0$ 和 $\beta_2\cos\theta-|a-y|\sin\theta>0$，所以当 $R\to\infty$ 和 $r\to 0$ 时，自公式(17)及 I_4 的相应的式子就可导得

$$I_{2,4}=\pm i\int_0^\infty\left\{e^{[\beta_1+i|a\mp y|]i\xi}-e^{[\beta_2+i|a\mp y|]i\xi}\right\}\frac{d\xi}{\xi^2}\mp i(\beta_2-\beta_1)\frac{\pi}{2}. \tag{18}$$

把公式(16)和(18)的四个式子相加，即得

$$J_5=\frac{1}{2}\int_0^\infty\left(e^{-|a+y|\xi}-e^{-|a-y|\xi}\right)\frac{\sin(\beta_1\xi)-\sin(\beta_2\xi)}{\xi^2}d\xi=$$

$$=\frac{1}{2}\left\{\frac{1}{2}\beta_1\ln\frac{\beta_1^2+|a-y|^2}{\beta_1^2+|a+y|^2}+|a-y|\operatorname{arc\,tg}\frac{\beta_1}{|a-y|}-\right.$$

$$\left.-|a+y|\operatorname{arc\,tg}\frac{\beta_1}{|a+y|}\right\}-\frac{1}{2}\left\{\frac{1}{2}\beta_2\ln\frac{\beta_2^2+|a-y|^2}{\beta_2^2+|a+y|^2}+\right.$$

$$\left.+|a-y|\operatorname{arc\,tg}\frac{\beta_2}{|a-y|}-|a+y|\operatorname{arc\,tg}\frac{\beta_2}{|a+y|}\right\}, \tag{19}$$

其中显示，各有关值的绝对值符号可以取消．把公式(14)和(19)代入公式(12)，即得边界值问题的解答：

$$
\left.
\begin{aligned}
\bar{\sigma}_x = & -\frac{GK}{\pi}\left\{\operatorname{arc\,tg}\frac{a+y}{c+b+x}-\operatorname{arc\,tg}\frac{a+y}{c-b+x}+\right. \\
& \left.+\operatorname{arc\,tg}\frac{a-y}{c+b+x}-\operatorname{arc\,tg}\frac{a-y}{c-b+x}\right\}- \\
& -\frac{2GKx}{\pi}\left\{\frac{a+y}{(a+y)^2+(c+b+x)^2}-\frac{a+y}{(a+y)^2+(c-b+x)^2}+\right. \\
& \left.+\frac{a-y}{(a-y)^2+(c+b+x)^2}-\frac{a-y}{(a-y)^2+(c-b+x)^2}\right\}, \\
\bar{\sigma}_y = & -\frac{3GK}{\pi}\left\{\operatorname{arc\,tg}\frac{a+y}{c+b+x}-\operatorname{arc\,tg}\frac{a+y}{c-b+x}+\right. \\
& \left.+\operatorname{arc\,tg}\frac{a-y}{c+b+x}-\operatorname{arc\,tg}\frac{a-y}{c-b+x}\right\}+ \\
& +\frac{2GKx}{\pi}\left\{\frac{a+y}{(a+y)^2+(c+b+x)^2}-\frac{a+y}{(a+y)^2+(c-b+x)^2}+\right. \\
& \left.+\frac{a-y}{(a-y)^2+(c+b+x)^2}-\frac{a-y}{(a-y)^2+(c-b+x)^2}\right\}, \\
\bar{\tau}_{xy} = & -\frac{2GKx}{\pi}\left\{\frac{c+b+x}{(a-y)^2+(c+b+x)^2}-\frac{c-b+x}{(a-y)^2+(c-b+x)^2}-\right. \\
& \left.-\frac{c+b+x}{(a+y)^2+(c+b+x)^2}+\frac{c-b+x}{(a+y)^2+(c-b+x)^2}\right\}+ \\
& +\frac{GK}{2\pi}\ln\frac{[(c-b+x)^2+(a-y)^2][(c+b+x)^2+(a+y)^2]}{[(c+b+x)^2+(a-y)^2][(c-b+x)^2+(a+y)^2]}, \\
\bar{\sigma}_z = & -\frac{4\nu GK}{\pi}\left\{\operatorname{arc\,tg}\frac{a+y}{c+b+x}-\operatorname{arc\,tg}\frac{a+y}{c-b+x}+\right. \\
& \left.+\operatorname{arc\,tg}\frac{a-y}{c+b+x}-\operatorname{arc\,tg}\frac{a-y}{c-b+x}\right\}, \\
\bar{v} = & -\frac{(3-4\nu)K}{4\pi}\left\{(c+b+x)\ln\frac{(c+b+x)^2+(a-y)^2}{(c+b+x)^2+(a+y)^2}-\right. \\
& -(c-b+x)\ln\frac{(c-b+x)^2+(a-y)^2}{(c-b+x)^2+(a+y)^2}+ \\
& +2(a-y)\left[\operatorname{arc\,tg}\frac{c+b+x}{a-y}-\operatorname{arc\,tg}\frac{c-b+x}{a-y}\right]- \\
& \left.-2(a+y)\left[\operatorname{arc\,tg}\frac{c+b+x}{a+y}-\operatorname{arc\,tg}\frac{c-b+x}{a+y}\right]\right\}+ \\
& +\frac{Kx}{2\pi}\ln\frac{[(c-b+x)^2+(a-y)^2][(c+b+x)^2+(a+y)^2]}{[(c+b+x)^2+(a-y)^2][(c-b+x)^2+(a+y)^2]}, \\
\bar{u} = & \frac{(3-4\nu)K}{4\pi}\left\{(a+y)\ln\frac{(a+y)^2+(c-b+x)^2}{(a+y)^2+(c+b+x)^2}+\right.
\end{aligned}
\right\} \quad (20)
$$

$$
\begin{aligned}
&+(a-y)\ln\frac{(a-y)^2+(c-b+x)^2}{(a-y)^2+(c+b+x)^2}+\\
&+2(c-b+x)\left[\operatorname{arc\,tg}\frac{a+y}{c-b+x}+\operatorname{arc\,tg}\frac{a-y}{c-b+x}\right]-\\
&-2(c+b+x)\left[\operatorname{arc\,tg}\frac{a+y}{c+b+x}+\operatorname{arc\,tg}\frac{a-y}{c+b+x}\right]\Bigg\}+\\
&+\frac{Kx}{\pi}\left[\operatorname{arc\,tg}\frac{a+y}{c+b+x}-\operatorname{arc\,tg}\frac{a+y}{c-b+x}+\right.\\
&\left.+\operatorname{arc\,tg}\frac{a-y}{c+b+x}-\operatorname{arc\,tg}\frac{a-y}{c-b+x}\right].
\end{aligned}
$$

三、半无限弹性組合体里的热应力和位移

把公式(3)至(8)跟公式(20)相加，并引进关系 $x=c\bar{x}$，$y=c\bar{y}$，$a=c\bar{a}$ 和 $b=c\bar{b}$，就得到本题的全部解：

$$
\begin{aligned}
\sigma_x=&-\frac{GK}{\pi}\Bigg\{\operatorname{arc\,tg}\frac{\bar{x}-1-\bar{b}}{\bar{y}-\bar{a}}-\operatorname{arc\,tg}\frac{\bar{x}-1+\bar{b}}{\bar{y}-\bar{a}}-\operatorname{arc\,tg}\frac{\bar{x}-1-\bar{b}}{\bar{y}+\bar{a}}+\\
&+\operatorname{arc\,tg}\frac{\bar{x}-1+\bar{b}}{\bar{y}+\bar{a}}+\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\\
&+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}+2\bar{x}\left[(\bar{a}+\bar{y})\left(\frac{1}{\bar{r}_7^2}-\frac{1}{\bar{r}_6^2}\right)+\right.\\
&\left.+(\bar{a}-\bar{y})\left(\frac{1}{\bar{r}_8^2}-\frac{1}{\bar{r}_5^2}\right)\right]\Bigg\},\\
\sigma_y=&-\frac{GK}{\pi}\Bigg\{\operatorname{arc\,tg}\frac{\bar{y}-\bar{a}}{\bar{x}-1-\bar{b}}-\operatorname{arc\,tg}\frac{\bar{y}+\bar{a}}{\bar{x}-1-\bar{b}}-\operatorname{arc\,tg}\frac{\bar{y}-\bar{a}}{\bar{x}-1+\bar{b}}+\\
&+\operatorname{arc\,tg}\frac{\bar{y}+\bar{a}}{\bar{x}-1+\bar{b}}+3\left[\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\right.\\
&\left.+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}\right]-2\bar{x}\left[(\bar{a}+\bar{y})\left(\frac{1}{\bar{r}_7^2}-\frac{1}{\bar{r}_6^2}\right)+\right.\\
&\left.+(\bar{a}-\bar{y})\left(\frac{1}{\bar{r}_8^2}-\frac{1}{\bar{r}_5^2}\right)\right]\Bigg\},\\
\tau_{xy}=&\frac{GK}{\pi}\left\{\ln\frac{\bar{r}_1\bar{r}_3\bar{r}_5\bar{r}_7}{\bar{r}_2\bar{r}_4\bar{r}_6\bar{r}_8}-2\bar{x}\left[(1+\bar{b}+\bar{x})\left(\frac{1}{\bar{r}_8^2}-\frac{1}{\bar{r}_7^2}\right)+(1-\bar{b}+\bar{x})\left(\frac{1}{\bar{r}_6^2}-\frac{1}{\bar{r}_5^2}\right)\right]\right\},\\
\sigma_{zi}=&-2GK-\frac{4\nu GK}{\pi}\left\{\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\right.\\
&\left.+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}\right\},\\
\sigma_{za}=&-\frac{4\nu GK}{\pi}\left\{\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\right.\\
&\left.+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}\right\},
\end{aligned}
$$

$$
\left.\begin{aligned}
v=&\frac{Kc}{2\pi}\Big\{(\bar{x}-1-\bar{b})\ln\frac{\bar{r}_1}{\bar{r}_2}+(\bar{x}-1+\bar{b})\ln\frac{\bar{r}_3}{\bar{r}_4}+\\
&+(\bar{y}-\bar{a})\left[\operatorname{arc\,tg}\frac{\bar{x}-1-\bar{b}}{\bar{y}-\bar{a}}-\operatorname{arc\,tg}\frac{\bar{x}-1+\bar{b}}{\bar{y}-\bar{a}}\right]-\\
&-(\bar{y}+\bar{a})\left[\operatorname{arc\,tg}\frac{\bar{x}-1-\bar{b}}{\bar{y}+\bar{a}}-\operatorname{arc\,tg}\frac{\bar{x}-1+\bar{b}}{\bar{y}+\bar{a}}\right]\Big\}-\\
&-\frac{(3-4\nu)Kc}{2\pi}\Big\{(1+\bar{b}+\bar{x})\ln\frac{\bar{r}_8}{\bar{r}_7}+(1-\bar{b}+\bar{x})\ln\frac{\bar{r}_6}{\bar{r}_5}+\\
&+(\bar{a}-\bar{y})\left[\operatorname{arc\,tg}\frac{1+\bar{b}+\bar{x}}{\bar{a}-\bar{y}}-\operatorname{arc\,tg}\frac{1-\bar{b}+\bar{x}}{\bar{a}-\bar{y}}\right]-\\
&-(\bar{a}+\bar{y})\left[\operatorname{arc\,tg}\frac{1+\bar{b}+\bar{x}}{\bar{a}+\bar{y}}-\operatorname{arc\,tg}\frac{1-\bar{b}+\bar{x}}{\bar{a}+\bar{y}}\right]\Big\}-\frac{Kc\bar{x}}{\pi}\ln\frac{\bar{r}_6\bar{r}_8}{\bar{r}_5\bar{r}_7},\\
u=&\frac{Kc}{2\pi}\Big\{(\bar{y}-\bar{a})\ln\frac{\bar{r}_1}{\bar{r}_4}-(\bar{y}+\bar{a})\ln\frac{\bar{r}_2}{\bar{r}_3}+\\
&+(\bar{x}-1-\bar{b})\left[\operatorname{arc\,tg}\frac{\bar{y}-\bar{a}}{\bar{x}-1-\bar{b}}-\operatorname{arc\,tg}\frac{\bar{y}+\bar{a}}{\bar{x}-1-\bar{b}}\right]-\\
&-(\bar{x}-1+\bar{b})\left[\operatorname{arc\,tg}\frac{\bar{y}-\bar{a}}{\bar{x}-1+\bar{b}}-\operatorname{arc\,tg}\frac{\bar{y}+\bar{a}}{\bar{x}-1+\bar{b}}\right]\Big\}+\\
&+\frac{(3-4\nu)Kc}{2\pi}\Big\{(\bar{a}+\bar{y})\ln\frac{\bar{r}_6}{\bar{r}_7}+(\bar{a}-\bar{y})\ln\frac{\bar{r}_5}{\bar{r}_8}+\\
&+(1-\bar{b}+\bar{x})\left[\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}\right]-\\
&-(1+\bar{b}+\bar{x})\left[\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}\right]\Big\}+\\
&+\frac{Kc}{\pi}\bar{x}\left[\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}+\bar{y}}{1-\bar{b}+\bar{x}}+\right.\\
&\left.+\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1+\bar{b}+\bar{x}}-\operatorname{arc\,tg}\frac{\bar{a}-\bar{y}}{1-\bar{b}+\bar{x}}\right],
\end{aligned}\right\}\tag{21}
$$

其中 σ_{zi} 和 σ_{za} 各代表矩形內外沿 z 轴的应力，并曾用了符号

$$
\begin{aligned}
&\bar{r}_1^2=(\bar{x}-1-\bar{b})^2+(\bar{y}-\bar{a})^2, && \bar{r}_2^2=(\bar{x}-1-\bar{b})^2+(\bar{y}+\bar{a})^2,\\
&\bar{r}_3^2=(\bar{x}-1+\bar{b})^2+(\bar{y}+\bar{a})^2, && \bar{r}_4^2=(\bar{x}-1+\bar{b})^2+(\bar{y}-\bar{a})^2,\\
&\bar{r}_5^2=(1-\bar{b}+\bar{x})^2+(\bar{a}-\bar{y})^2, && \bar{r}_6^2=(1-\bar{b}+\bar{x})^2+(\bar{a}+\bar{y})^2,\\
&\bar{r}_7^2=(1+\bar{b}+\bar{x})^2+(\bar{a}+\bar{y})^2, && \bar{r}_8^2=(1+\bar{b}+\bar{x})^2+(\bar{a}-\bar{y})^2.
\end{aligned}
$$

四、例題与討論

选用两种铁镍合金，其主要性态是：$E=E_i=E_a=2\times10^6\,\mathrm{kg/cm^2}$, $\alpha_i=5\times10^{-6}\,1/℃$, $\alpha_a=2\times10^{-6}\,1/℃$, $\nu=\nu_i=\nu_a=0.3$, $\eta_0=\alpha_i-\alpha_a=3\times10^{-6}1/℃$, $G=E/2(1+\nu)=7.7\times10^5\,\mathrm{kg/cm^2}$; 取 $GK=1000\,\mathrm{kg}$, 则 $T_0=233℃$, $K=\dfrac{1+\nu}{1-\nu}\eta_0T_0=1.298\times10^{-3}$. 又设 $c=10\,\mathrm{cm}$, $\bar{a}=0.5$, $\bar{b}=0.25$. 为了照顾到各物理量的连续性和非连续性明显出

现，取 $\bar{y}=0,\ 0.25,\ 0.4,\ 0.5,\ 0.6,\ 0.75,\ 1,\ 1.5$ 和 $\bar{x}=0,\ 0.5,\ 0.75,\ 1,\ 1.25,\ 1.5$.

数字计算按照图 3 的各点进行，用横竖线标志其所属的坐标距离．把数字计算得出的结果绘成图 4 至图 9，其中实线表示矩形以外的物理量，虚线表示矩形里边的物理量．

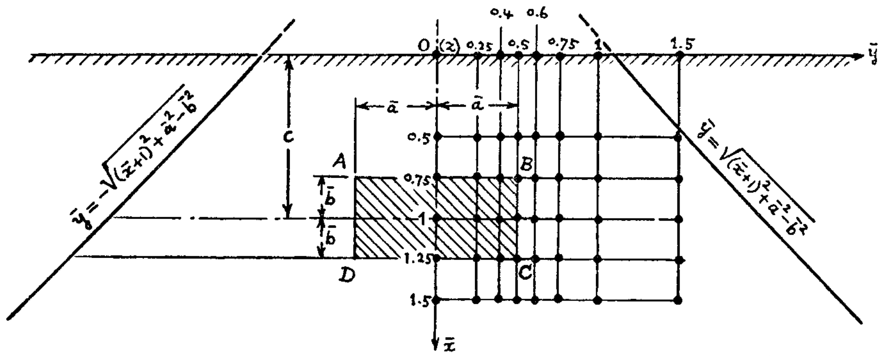

图 3　数字计算点分布图；$\sigma_{za}=0$ 的双曲线曲面的轨迹

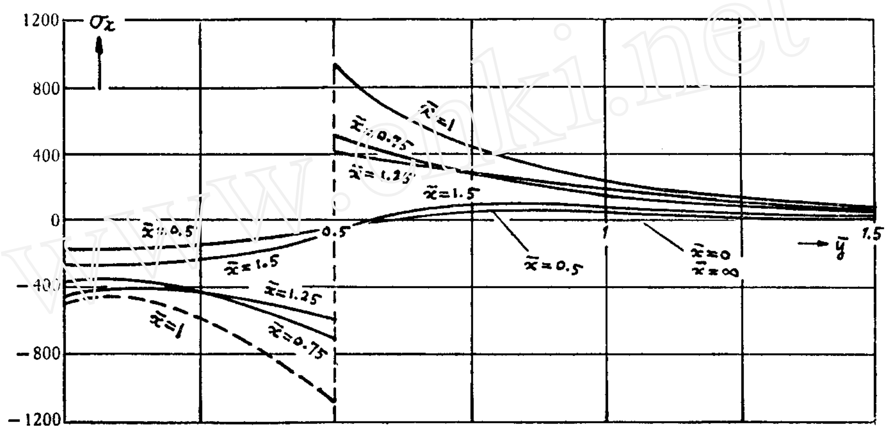

图 4　应力 σ_x 的变化

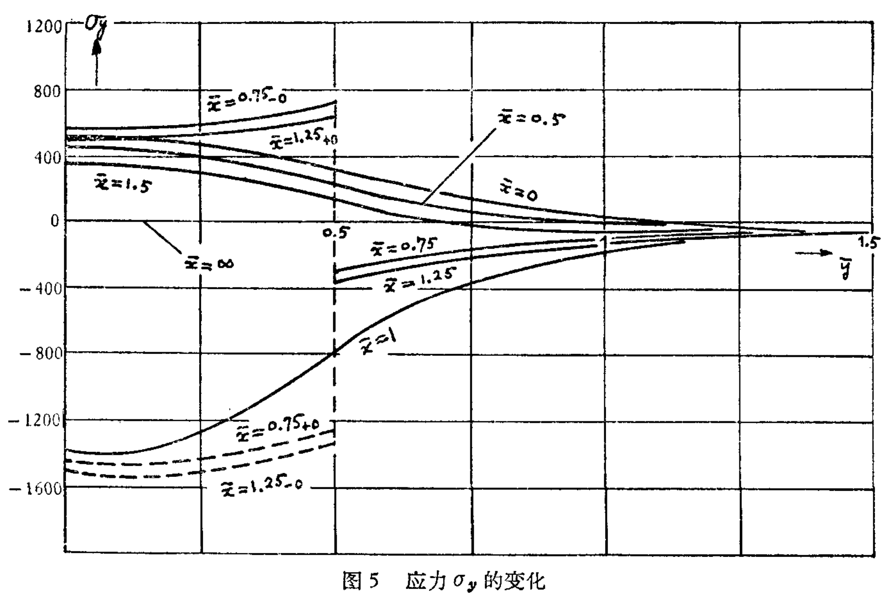

图 5　应力 σ_y 的变化

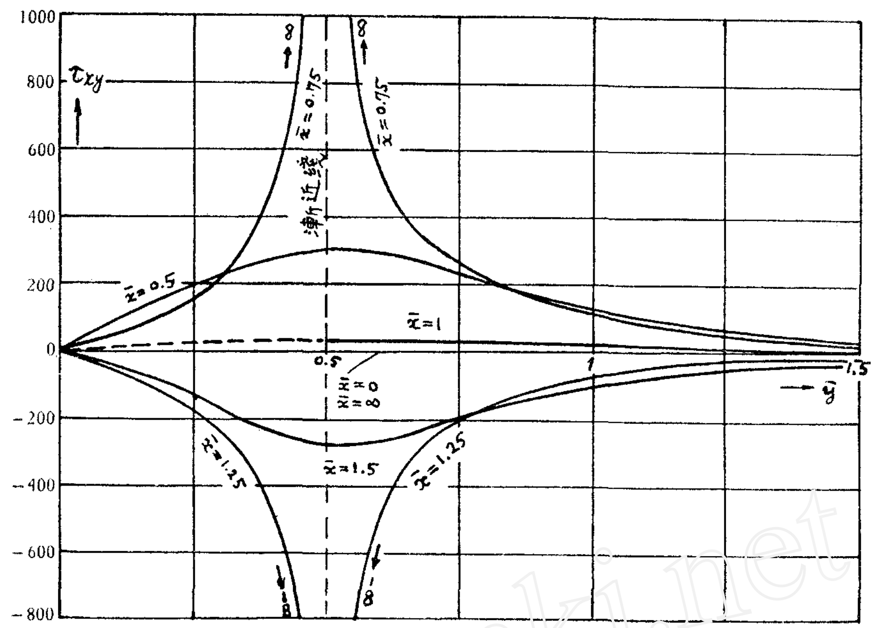

图 6 应力 τ_{xy} 的变化

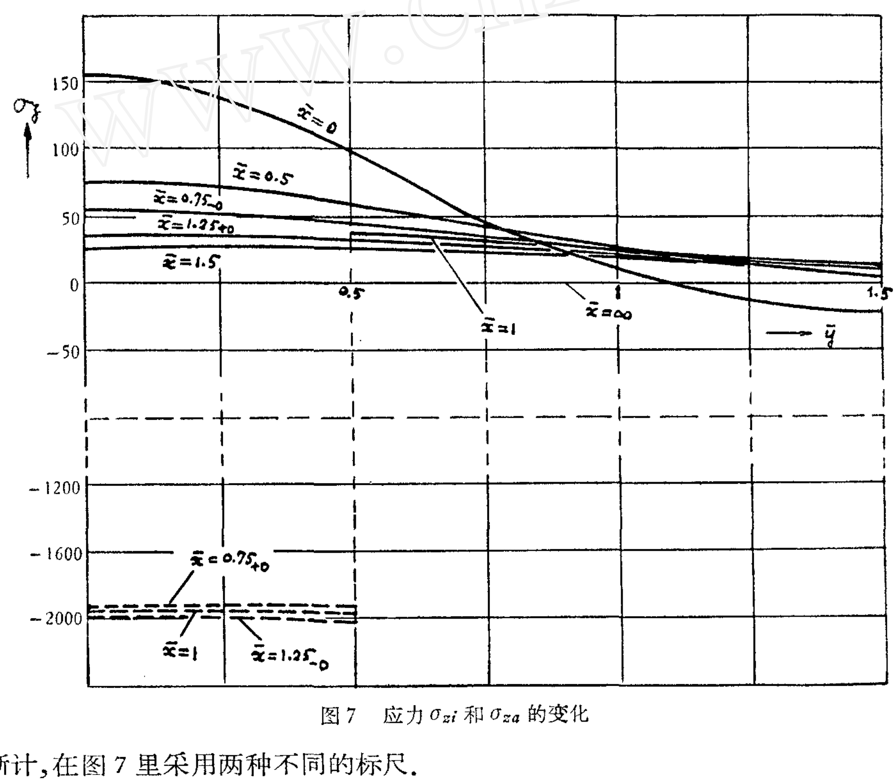

图 7 应力 σ_{zi} 和 σ_{za} 的变化

为清晰计,在图 7 里采用两种不同的标尺.

首先,图 4 至图 9 显示:当 σ_x 和 σ_y 垂直于分界线时,它们是连续的;当它们平行于分界线时,它们是非连续的;σ_z 在分界线上都是非连续的,因为 σ_z 总是平行于分界面;在矩形的角上,τ_{xy} 变为无限大;在分界线上,v 和 u 都连续的. 图中曲线所显示的连续性

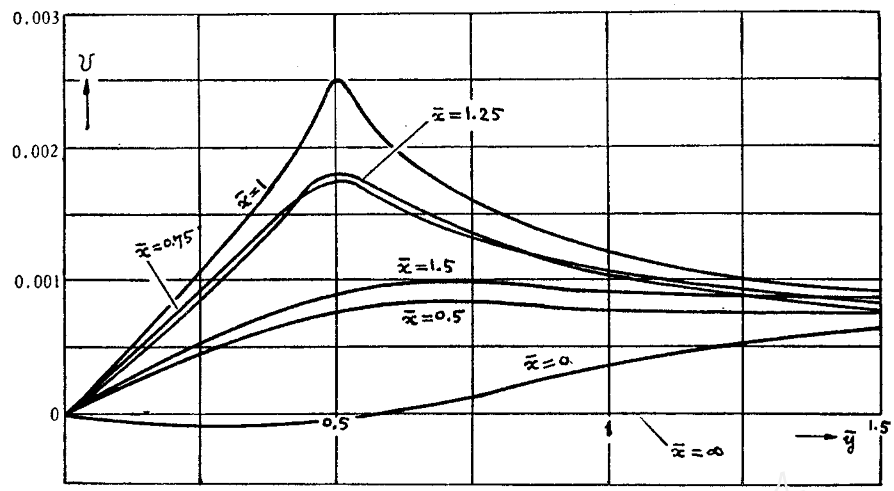

图 8　位移 v 的变化

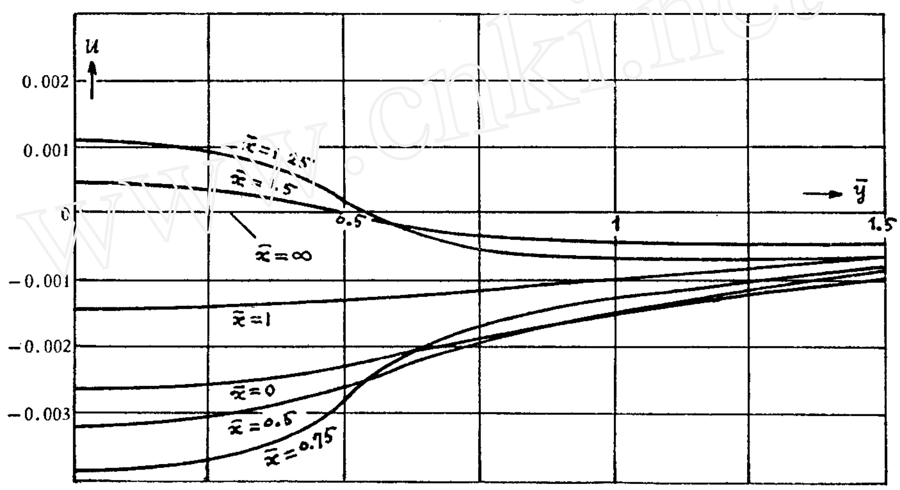

图 9　位移 u 的变化

和非连续性情况均与有关的原理符合[3].

其次，倘须用分析来肯定一些为曲线所不易规定的细微性态．可以验证：$\left(\frac{\partial}{\partial\bar{y}}\sigma_x\right)_{\substack{\bar{y}=0\\\bar{x}=0}} = \left(\frac{\partial}{\partial\bar{y}}\sigma_y\right)_{\substack{\bar{y}=0\\\bar{x}=0}} = \left(\frac{\partial}{\partial\bar{y}}\tau_{xy}\right)_{\substack{\bar{y}=0\\\bar{x}=0}} = 0$，$\left(\frac{\partial}{\partial\bar{y}}\tau_{xy}\right)_{\substack{\bar{y}=0\\\bar{x}=0}} \gtreqless 0$，$\left(\frac{\partial}{\partial\bar{y}}\sigma_{zi}\right)_{\bar{y}=0} = \left(\frac{\partial}{\partial\bar{y}}\sigma_{za}\right)_{\bar{y}=0} = 0$，$\left(\frac{\partial v}{\partial\bar{y}}\right)_{\bar{y}=0} \gtreqless 0$，$\left(\frac{\partial u}{\partial\bar{y}}\right)_{\bar{y}=0} = 0$；由于

$$\left(\frac{\partial}{\partial\bar{y}}\sigma_x\right)_{\substack{\bar{y}=0\\\bar{x}\neq0}} = -\left(\frac{\partial}{\partial\bar{y}}\sigma_y\right)_{\substack{\bar{y}=0\\\bar{x}\neq0}} = -\frac{4GK\bar{x}}{\pi}\left(\frac{1+\bar{b}+\bar{x}}{\bar{a}^2+(1+\bar{b}+\bar{x})^2}+\right.$$
$$\left.+\frac{1-\bar{b}+\bar{x}}{\bar{a}^2+(1-\bar{b}+\bar{x})^2}\right)\frac{f(\bar{a},\bar{b},\bar{x})}{[\bar{a}^2+(1+\bar{b}+\bar{x})^2][\bar{a}^2+(1-\bar{b}+\bar{x})^2]},$$

其中

$$f(\bar{a},\bar{b},\bar{x}) = (1+\bar{b}+\bar{x})[\bar{a}^2+(1-\bar{b}+\bar{x})^2]-$$
$$-(1-\bar{b}+\bar{x})[\bar{a}^2+(1+\bar{b}+\bar{x})^2] = 2\bar{b}[\bar{a}^2+\bar{b}^2-(1+\bar{x})^2],$$

所以

$$\left(\frac{\partial \sigma_x}{\partial \bar{y}}\right)_{\substack{\bar{y}=0\\ \bar{x}>0}} = -\left(\frac{\partial \sigma_y}{\partial \bar{y}}\right)_{\substack{\bar{y}=0\\ \bar{x}>0}} \gtreqless 0, \text{ 当 } \bar{a}^2+\bar{b}^2 \lesseqgtr (1+\bar{x})^2 \text{ 时}.$$

再次，条件 $\sigma_{xa}=0$ 能给出双曲线 $\bar{y}=\pm\sqrt{(\bar{x}+1)^2+\bar{a}^2-\bar{b}^2}$；已把这两条曲线绘于图 3；在点 $(-1, \pm\sqrt{\bar{a}^2-\bar{b}^2})$ 处，双曲线与直线 $\bar{y}=\pm\sqrt{\bar{a}^2-\bar{b}^2}$ 相切；双曲线的渐近线是 $\bar{y}=\pm(\bar{x}+1)$.

五、应力和位移在分界綫上的連續性

关于 σ_z 在分界线上的连续性，可见公式(6). 由于 τ_{xy}, v 和 u 的各项系显明的连续函数，所以不用检查.

根据原理，纯弹性力学边界值问题所给出的解都连续地跨越分界线，所以此地只需要对 $\bar{\sigma}_x$ 和 $\bar{\sigma}_y$ 进行检查.

因为在 AB 和 CD 分界线段上有

$$(\bar{\sigma}_x)_{x=c-b\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{-2b\mp\delta}{y-a} - \text{arc tg}\,\frac{\mp\delta}{y-a} - \text{arc tg}\,\frac{-2b\mp\delta}{y+a} + \text{arc tg}\,\frac{\mp\delta}{y+a}\right]$$

和

$$(\bar{\sigma}_x)_{x=c+b\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{\mp\delta}{y-a} - \text{arc tg}\,\frac{2b\mp\delta}{y-a} - \text{arc tg}\,\frac{\mp\delta}{y+a} + \text{arc tg}\,\frac{2b\mp\delta}{y+a}\right],$$

所以导出

$$(\sigma_x)_{x=c\mp b\pm 0} - (\sigma_x)_{x=c\mp b\mp 0} = 0; \tag{22}$$

更因在 BC 和 AD 两段分界线上有

$$(\bar{\sigma}_x)_{y=a\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{x-c-b}{\mp\delta} - \text{arc tg}\,\frac{x-c+b}{\mp\delta} - \right.$$
$$\left. - \text{arc tg}\,\frac{x-c-b}{2a\mp\delta} + \text{arc tg}\,\frac{x-c+b}{2a\mp\delta}\right]$$

和

$$(\bar{\sigma}_x)_{y=-a\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{x-c-b}{-2a\mp\delta} - \text{arc tg}\,\frac{x-c+b}{-2a\mp\delta} - \right.$$
$$\left. - \text{arc tg}\,\frac{x-c-b}{\mp\delta} + \text{arc tg}\,\frac{x-c+b}{\mp\delta}\right],$$

所以导得

$$(\sigma_x)_{y=\pm a\mp 0} - (\sigma_x)_{y=\pm a\pm 0} = -\frac{E\eta_0 T_0}{1-\nu}; \tag{23}$$

又因在 AB 和 CD 分界线段上有

$$(\bar{\sigma}_y)_{x=c-b\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{y-a}{-2b\mp\delta} - \text{arc tg}\,\frac{y+a}{-2b\mp\delta} - \text{arc tg}\,\frac{y-a}{\mp\delta} + \text{arc tg}\,\frac{y+a}{\mp\delta}\right]$$

和

$$(\bar{\sigma}_y)_{x=c+b\mp\delta} = -\frac{GK}{\pi}\left[\text{arc tg}\,\frac{y-a}{\mp\delta} - \text{arc tg}\,\frac{y+a}{\mp\delta} - \text{arc tg}\,\frac{y-a}{2b\mp\delta} + \text{arc tg}\,\frac{y+a}{2b\mp\delta}\right],$$

所以得出

$$(\sigma_y)_{x=c\mp b\pm 0} - (\sigma_y)_{x=c\mp b\mp 0} = -\frac{E\eta_0 T_0}{1-\nu}; \tag{24}$$

复因在 BC 和 AD 分界线上有

$$(\bar{\sigma}_y)_{y=a\mp\delta}=-\frac{GK}{\pi}\left[\operatorname{arc\,tg}\frac{\mp\delta}{x-c-b}-\operatorname{arc\,tg}\frac{2a\mp\delta}{x-c-b}-\right.$$
$$\left.-\operatorname{arc\,tg}\frac{\mp\delta}{x-c+b}+\operatorname{arc\,tg}\frac{2a\mp\delta}{x-c+b}\right]$$

和

$$(\bar{\sigma}_y)_{y=-a\mp\delta}=-\frac{GK}{\pi}\left[\operatorname{arc\,tg}\frac{-2a\mp\delta}{x-c-b}-\operatorname{arc\,tg}\frac{\mp\delta}{x-c-b}-\right.$$
$$\left.-\operatorname{arc\,tg}\frac{-2a\mp\delta}{x-c+b}+\operatorname{arc\,tg}\frac{\mp\delta}{x-c+b}\right],$$

所以得到

$$(\sigma_y)_{y=\pm a\mp 0}-(\sigma_y)_{y=\pm a\pm 0}=0. \tag{25}$$

公式(22)至(25)表示的连续性或突跃性都与有关的原理符合[3]。

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WÄRMESPANNUNGEN UND VERSCHIEBUNGEN IM HALBRAUM MIT RECHTECKIGEM EINSCHLUSS

LIU HSIEN-CHIH

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