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彈塑性圓板大撓度問題*

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一. 引　言

關於彈塑性板的小撓度問題已有不少工作，如 B. B. Соколовский[1]、D. Trifan[2]、A. C. Григорьев[3] 及 A. A. Ильюшин[4] 等，但關於彈塑性板的大撓度問題的工作則爲數不多，據作者們所知，有 P. M. Naghdi[5] 及 Ю. P. Лепик[6]. P. M. Naghdi 採用數值積分方法，運算甚爲繁複；Ю. P. Лепик 在得出基本方程後，簡略地討論了如何用彈性解近似法和變分法求方程的解，但未得出具體結果.

本文採用小彈塑性形變理論，考慮材料可壓縮性，求得以中心面之位移爲未知量的方程；當廣義應力-應變關係可表爲冪級數時，本文用載荷小參數法求得簡潔的結果；當廣義應力-應變關係表示爲一般函數時，本文提供一個逐次漸近解法.

二. 基 本 方 程

討論在均佈荷重下周邊固定的情形. 將主要記號列出如下:

u, w ——中心面徑向位移分量及撓度，向外向下爲正.

e_r, e_t ——任一點徑向、切向應變.

r, z ——任一點坐標，r 自中心向外計算，z 自中心面向下計算.

N_r, N_t ——徑向及切向張力.

M_r, M_t——徑向及切向彎矩，使下側受拉爲正.

Q_r ——環向斷面上的橫切力，內側向下爲正.

q ——均佈荷重，向下爲正.

σ_i, e_i ——廣義應力及廣義應變.

h ——半板厚.

R ——板的半徑.

本問題有下列幾類方程.

(1) 平衡方程. 實心圓板在均佈荷重作用下考慮大撓度的平衡方程爲[7]

$$\frac{d}{dr}(rN_r) - N_t = 0, \frac{d}{dr}(rM_r) - M_t + N_r \cdot r\frac{dw}{dr} + \frac{qr^2}{2} = 0. \qquad (1)$$

(2) 應變和位移關係. 根據法綫不變的假定有

* 本文兩位作者曾就同一題目分別於 1957 年 11 月 12 日及 27 日投稿本學報，因內容相似，故重新合寫成此文.

$$e_r = -w''z + u' + \frac{1}{2}w'^2,\ e_t = -\frac{w'}{r}z + \frac{u}{r}, \tag{2}$$

其中

$$u' = \frac{du}{dr},\ w'' = \frac{d^2w}{dr^2}, \cdots \tag{3}$$

以後仿此.

(3) 應力和應變關係. 按小彈塑性形變理論,不考慮減載現象有[4]

$$\left.\begin{aligned} \sigma_r &= \left(K - \frac{2}{9}\frac{\sigma_i}{e_i}\right)\theta + \frac{2\sigma_i}{3e_i}e_r, \\ \sigma_r &= \left(K - \frac{2}{9}\frac{\sigma_i}{e_i}\right)\theta + \frac{2\sigma_i}{3e_i}e_t, \\ \sigma_z &= \left(K - \frac{2}{9}\frac{\sigma_i}{e_i}\right)\theta + \frac{2\sigma_i}{3e_i}e_z, \end{aligned}\right\} \tag{4}$$

其中K爲體積改變常數, θ 爲體積變形, σ_i, e_i 爲廣義應力及應變:

$$\left.\begin{aligned} \theta &= e_r + e_t + e_z, \\ \sigma_i &= \sqrt{\sigma_r^2 - \sigma_r\sigma_t + \sigma_t^2}, \\ e_i &= \frac{\sqrt{2}}{3}\sqrt{(e_r - e_t)^2 + (e_t - e_z)^2 + (e_z - e_r)^2}. \end{aligned}\right\} \tag{5}$$

由 $\sigma_z = 0$, 根據(4)的末式得 e_z, 代入前二式消去 θ 得

$$\sigma_r = \phi(e_r + \alpha e_t),\ \sigma_t = \phi(e_t + \alpha e_r), \tag{6}$$

其中

$$\alpha = \frac{K - \frac{2}{9}\frac{\sigma_i}{e_i}}{2K + \frac{2\sigma_i}{9e_i}}, \quad \phi = \frac{2\sigma_i}{3e_i}\frac{2K + \frac{2}{9}\frac{\sigma_i}{e_i}}{K + \frac{4}{9}\frac{\sigma_i}{e_i}}. \tag{7}$$

設

$$\frac{\sigma_i}{e_i} = 3G[1 - \omega(e_i)], \tag{8}$$

並注意

$$K = \frac{E}{3(1-2\nu)},\ G = \frac{E}{2(1+\nu)}. \tag{9}$$

其中E爲彈性模數, ν 爲泊松比. 將(8), (9)代入(7), 得

$$\left.\begin{aligned} \alpha &= \frac{(1+\nu) - (1-2\nu)[1-\omega(e_i)]}{2(1+\nu) + (1-2\nu)[1-\omega(e_i)]}, \\ \phi &= \frac{E}{1+\nu}[1-\omega(e_i)]\frac{2(1+\nu) + (1-2\nu)[1-\omega(e_i)]}{(1+\nu) + 2(1-2\nu)[1-\omega(e_i)]}. \end{aligned}\right\} \tag{10}$$

$\omega(e_i)$ 爲 e_i 的函數,彈性時爲零,塑性變形很大時接近於 1. α 和 ϕ 有下述性質: 彈性時 $\alpha = \nu,\ \phi = \frac{E}{1-\nu^2}$, 塑性變形較大時 $\alpha \approx \frac{1}{2},\ \phi \approx \frac{2E}{1+\nu}[1-\omega(e_i)]$, 或

$$\nu \leqslant \alpha \leqslant \frac{1}{2},\ \frac{E}{1-\nu^2} \leqslant \phi \leqslant \frac{2E}{1+\nu}[1-\omega(e_i)]. \tag{11}$$

若材料不可壓縮，即 $\nu=\dfrac{1}{2}$，則恆有

$$\alpha=\frac{1}{2},\ \phi=\frac{2E}{1+\nu}[1-\omega(e_i)]=\frac{4E}{3}[1-\omega(e_i)]. \tag{12}$$

P. A. Межлумян[8] 求得的横向變形係數和(10)式的 α 相同.

以下將上述三類方程(1)，(2)，(6)進行代換和簡化．將(2)代入(6)，得

$$\left.\begin{aligned}\sigma_r&=-w''\phi z+\left(u'+\frac{1}{2}w'^2\right)\phi-\frac{w'}{r}\phi\alpha z+\frac{u}{r}\phi\alpha,\\ \sigma_t&=-\frac{w'}{r}\phi z+\frac{u}{r}\phi-w''\phi\alpha z+\left(u'+\frac{1}{2}w'^2\right)\phi\alpha.\end{aligned}\right\} \tag{13}$$

由(13)得

$$\left.\begin{aligned}N_r&=\int_{-h}^{+h}\sigma_r dz=-w''J_2+\left(u'+\frac{1}{2}w'^2\right)J_1-\frac{w'}{r}I_2+\frac{u}{r}I_1,\\ N_t&=\int_{-h}^{+h}\sigma_t dz=-\frac{w'}{r}J_2+\frac{u}{r}J_1-w''I_2+\left(u'+\frac{1}{2}w'^2\right)I_1;\\ M_r&=\int_{-h}^{+h}\sigma_r z dz=-w''J_3+\left(u'+\frac{1}{2}w'^2\right)J_2-\frac{w'}{r}I_3+\frac{u}{r}I_2,\\ M_t&=\int_{-h}^{+h}\sigma_t z dz=-\frac{w'}{r}J_3+\frac{u}{r}J_2-w''I_3+\left(u'+\frac{1}{2}w'^2\right)I_2;\end{aligned}\right\} \tag{14}$$

其中 h 為板之半高．

$$\left.\begin{aligned}&J_1=\int_{-h}^{+h}\phi dz,\ J_2=\int_{-h}^{+h}\phi z dz,\ J_3=\int_{-h}^{+h}\phi z^2 dz;\\ &I_1=\int_{-h}^{+h}\phi\alpha dz,\ I_2=\int_{-h}^{+h}\phi\alpha z dz,\ I_3=\int_{-h}^{+h}\phi\alpha z^2 dz.\end{aligned}\right\} \tag{15}$$

將(14)代入方程(1)，得

$$\left.\begin{aligned}&-\left(w'''r+w''-\frac{w'}{r}\right)J_2+\left(u''r+u'-\frac{u}{r}+\frac{1}{2}w'^2+rw'w''\right)J_1-\\ &\quad-\frac{1}{2}w'^2I_1-w''rJ_2'-w'I_2'+\left(ru'+\frac{1}{2}rw'^2\right)J_1'+uI_1'=0,\\ &-\left(w'''r+w''-\frac{w'}{r}\right)J_3+\left(u''r+u'-\frac{u}{r}+\frac{1}{2}w'^2\right)J_2-\frac{3}{2}w'^2I_2+\\ &\quad+\left(rw'u'+\frac{1}{2}rw'^3\right)J_1+w'uI_1-rw''J_3'+\left(ru'+\frac{1}{2}rw'^2\right)J_2'-\\ &\quad-w'I_3'+uI_2'+\frac{qr^2}{2}=0.\end{aligned}\right\} \tag{16}$$

給定 σ_i-e_i 關係或 $\omega(e_i)$，則通過(15)，(10)，(2)等，可將 I,J 表成 w,u 之函數；從而由(16)得到 w,u 的方程．

三．廣義應力-應變關係用冪級數表示時的解

D. Trifan 曾提出下列形式之 σ_i-e_i 關係：

$$\sigma_i = 3G(1-\gamma_1 e_i^2 + \cdots)e_i, \tag{17}$$

其中 $\gamma_1\cdots$ 爲常數，他曾僅取括號內前兩項與 24 S-T 鋁合金的試驗結果比較，在一定範圍內相當符合[2]。

今取(17)中括號內前兩項，並與(8)相比，知 $\omega(e_i)=\gamma_1 e_i^2$. 代入(10)，注意到有關 e_i 的項較小，將其展成 e_i 的冪級數：

$$\left.\begin{aligned}\alpha &= \nu+\nu_1 e_i^2+\cdots, & \nu_1 &= \frac{(1+\nu)(1-2\nu)}{3}\gamma_1,\\ \phi &= \frac{E}{1-\nu^2}(1-a_1 e_i^2+\cdots), & a_1 &= \frac{2}{3}\frac{1-\nu+\nu^2}{1-\nu}\gamma_1.\end{aligned}\right\} \tag{18}$$

因此

$$\phi\alpha = \frac{E}{1-\nu^2}(\nu+b_1 e_i^2+\cdots),\quad b_1=\frac{1-4\nu+\nu^2}{3(1-\nu)}\gamma_1. \tag{19}$$

上兩式中的 e_i，爲簡化起見，可用體積不變的結果，即設 $\theta=0$，由(5)得

$$e_i = \frac{2}{\sqrt{3}}\sqrt{e_r^2+e_re_t+e_t^2}. \tag{20}$$

前面我們沒有設體積不變，嚴格地說不能用(20)式。但事實上(18)，(19)中 e_i^2 項僅當 e_i 較大時才有顯著影響，而當變形較大時可以認爲體積不變。

將(2)代入(20)得

$$e_i = \frac{2}{\sqrt{3}}\sqrt{E_1z^2+E_2z+E_3}, \tag{21}$$

其中

$$\left.\begin{aligned}E_1 &= w''^2+\frac{w'^2}{r^2}+w''\frac{w'}{r},\\ E_2 &= -2u'w''-w'^2w''-2\frac{uw'}{r^2}-\frac{1}{r}u'w'-\frac{1}{2}\frac{w'^3}{r}-\frac{uw''}{r},\\ E_3 &= u'^2+\frac{1}{4}w'^4+\frac{u^2}{r^2}+u'w'^2+\frac{u}{r}u'+\frac{1}{2}\frac{uw'^2}{r}.\end{aligned}\right\} \tag{22}$$

將(21)代入(18)，(19)，取前兩項，即得

$$\left.\begin{aligned}\alpha &= \nu+\frac{4\nu_1}{3}(E_1z^2+E_2z+E_3),\\ \phi &= \frac{E}{1-\nu^2}\left[1-\frac{4a_1}{3}(E_1z^2+E_2z+E_3)\right],\\ \phi\alpha &= \frac{E}{1-\nu^2}\left[\nu+\frac{4b_1}{3}(E_1z^2+E_2z+E_3)\right].\end{aligned}\right\} \tag{23}$$

由(15)得

$$\left.\begin{aligned}J_1 &= \frac{E}{1-\nu^2}\left[2h-\frac{8}{9}a_1h^3E_1-\frac{8}{3}a_1hE_3\right], & I_1 &= \frac{E}{1-\nu^2}\left[2\nu h+\frac{8}{9}b_1h^3E_1+\frac{8}{3}b_1hE_3\right],\\ J_2 &= \frac{E}{1-\nu^2}\left[-\frac{8}{9}a_1h^3E_2\right], & I_2 &= \frac{E}{1-\nu^2}\left[\frac{8}{9}b_1h^3E_2\right],\\ J_3 &= \frac{E}{1-\nu^2}\left[\frac{2}{3}h^3-\frac{8}{15}a_1h^5E_1-\frac{8}{9}a_1h^3E_3\right], & I_3 &= \frac{E}{1-\nu^2}\left[\frac{2}{3}\nu h^3+\frac{8}{15}b_1h^5E_1+\frac{8}{9}b_1h^3E_3\right].\end{aligned}\right\} \tag{24}$$

以後採用無量綱量 $\bar{w}, \bar{u}, \bar{z}, \cdots$ 等:

$$\left.\begin{aligned} &w = \bar{w}h, \quad u = \frac{h^2}{R}\bar{u}, \quad r = R\bar{r}, \quad z = h\bar{z}, \quad q = \frac{Eh^4}{(1-\nu^2)R^4}\bar{q}, \\ &E_n = \frac{h^{n+1}}{R^4}\bar{E}_n, \quad J_n = \frac{Eh^n}{(1-\nu^2)}\bar{J}_n, \quad I_n = \frac{Eh^n}{(1-\nu^2)}\bar{I}_n, \quad \phi = \frac{E}{1-\nu^2}\bar{\phi}, \\ &\sigma = \frac{Eh^2}{(1-\nu^2)R^2}\bar{\sigma}, \quad M = \frac{Eh^4}{(1-\nu^2)R^2}\bar{M}, \quad N = \frac{Eh^3}{(1-\nu^2)R^2}\bar{N}, \end{aligned}\right\} \tag{25}$$

其中 σ, M, N 的附標略去.

(13),(14),(16),(22)等式中諸量加記號"—"仍成立. 為了簡化起見,略去此記號,以後推導各式皆指無量綱式. (23),(24)的無量綱式為

$$\left.\begin{aligned} \alpha &= \nu + \frac{4\nu_1 h^4}{3R^4}(E_1 z^2 + E_2 z + E_3), \\ \phi &= 1 - \frac{4a_1 h^4}{3R^4}(E_1 z^2 + E_2 z + E_3), \\ \phi\alpha &= \nu + \frac{4b_1 h^4}{3R^4}(E_1 z^2 + E_2 z + E_3); \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} &J_1 = 2 - \frac{8h^4}{9R^4}a_1E_1 - \frac{8h^4}{3R^4}a_1E_3, && I_1 = 2\nu + \frac{8h^4}{9R^4}b_1E_1 + \frac{8h^4}{3R^4}b_1E_3, \\ &J_2 = -\frac{8h^4}{9R^4}a_1E_2, && I_2 = \frac{8h^4}{9R^4}b_1E_2, \\ &J_3 = \frac{2}{3} - \frac{8h^4}{15R^4}a_1E_1 - \frac{8h^4}{9R^4}a_1E_3, && I_3 = \frac{2}{3}\nu + \frac{8h^4}{15R^4}b_1E_1 + \frac{8h^3}{9R}b_1E_3. \end{aligned}\right\} \tag{27}$$

(22)和(27)給出了 I 和 J,因此代入基本方程(16)就可以得到 w, u 的方程.

用荷載小參數法求解,設

$$w = w_1 q + w_3 q^3 + \cdots, \quad u = u_2 q^2 + u_4 q^4 + \cdots, \tag{28}$$

其中 w_i, u_i 為 r 的函數. 將上式代入(16),並注意 I, J 由(27),(22)給定,歸併 q 的同次項,令各項為零,可得關於 w_i, u_i 的一系列方程.

由 q 的一次項得方程

$$\frac{2}{3}\left(r w_1''' + w_1'' - \frac{w_1'}{r}\right) = \frac{r^2}{2}. \tag{29}$$

設圓板周邊固定,邊界條件為

$$w_1\Big|_{r=0} = 有限, \qquad w_1\Big|_{r=1} = 0, \qquad w_1'\Big|_{r=1} = 0, \tag{30}$$

解得

$$w_1 = \frac{3}{128}(r^2-1)^2. \tag{31}$$

由 q 的二次項可得關於 u_2, w_1 的方程. 將(31)代入得

$$r u_2'' + u_2' - \frac{u_2}{r} = \frac{9(\nu-7)}{2\cdot 32^2}r^5 - \frac{9(\nu-5)}{32^2}r^3 + \frac{9(\nu-3)}{2\cdot 32^2}r. \tag{32}$$

邊界條件為

$$u_2\Big|_{r=0}=\text{有限},\qquad u_2\Big|_{r=1}=0,\tag{33}$$

解得

$$u_2=\frac{3}{32^3}[(\nu-7)r^7-4(\nu-5)r^5+6(\nu-3)r^3-(3\nu-5)r].\tag{34}$$

由 q 的三次項得關於 w_3, w_1, u_2 的方程，將 w_1, u_2 代入即得

$$rw_3'''+w_3''-\frac{w_3'}{r}=\frac{\alpha_1}{10}r^{10}+\frac{\alpha_2}{8}r^8+\frac{\alpha_3}{6}r^6+\frac{\alpha_4}{4}r^4+\frac{\alpha_5}{2}r^2+$$
$$+\frac{4h^4}{3R^4}\left(\frac{\alpha_1^*}{6}r^6+\frac{\alpha_2^*}{4}r^4+\frac{\alpha_3^*}{2}r^2\right),\tag{35}$$

其中

$$\left.\begin{aligned}
&\alpha_1=\frac{27}{32^4}(11\nu^2-7\nu-10), && \alpha_1^*=-\frac{3^5}{20\cdot 32^2}(13b_1-65a_1),\\
&\alpha_2=-\frac{27}{32^4}(45\nu^2-27\nu-40), && \alpha_2^*=\frac{3^4}{10\cdot 32^2}(19b_1-55a_1),\\
&\alpha_3=\frac{27}{32^4}(70\nu^2-38\nu-60), && \alpha_3^*=\frac{3^5}{10\cdot 32^2}(b_1-2a_1).\\
&\alpha_4=-\frac{27}{32^4}(45\nu^2-31\nu-44),\\
&\alpha_5=\frac{27}{32^4}(9\nu^2-9\nu-10),
\end{aligned}\right\}\tag{36}$$

邊界條件為

$$w_3\Big|_{r=0}=\text{有限},\qquad w_3\Big|_{r=1}=0,\qquad w_3'\Big|_{r=1}=0,\tag{37}$$

解得

$$w_3=\left(\frac{\alpha_1}{10^2\cdot 12^2}r^{12}+\frac{\alpha_2}{8^2\cdot 10^2}r^{10}+\frac{\alpha_3}{6^2\cdot 8^2}r^8+\frac{\alpha_4}{4^2\cdot 6^2}r^6+\frac{\alpha_5}{2^2\cdot 4^2}r^4+\frac{\alpha_6}{4}r^2+\alpha_7\right)+$$
$$+\frac{4h^4}{3R^4}\left(\frac{\alpha_1^*}{6^2\cdot 8^2}r^8+\frac{\alpha_2^*}{4^2\cdot 6^2}r^6+\frac{\alpha_3^*}{2^2\cdot 4^2}r^4+\frac{\alpha_4^*}{4}r^2+\alpha_5^*\right),\tag{38}$$

其中

$$\left.\begin{aligned}
&\alpha_6=-\left(\frac{2\alpha_1}{10^2\cdot 12}+\frac{2\alpha_2}{8^2\cdot 10}+\frac{2\alpha_3}{6^2\cdot 8}+\frac{2\alpha_4}{4^2\cdot 6}+\frac{2\alpha_5}{2^2\cdot 4}\right),\\
&\alpha_7=\frac{5\alpha_1}{10^2\cdot 12^2}+\frac{4\alpha_2}{8^2\cdot 10^2}+\frac{3\alpha_3}{6^2\cdot 8^2}+\frac{2\alpha_4}{4^2\cdot 6^2}+\frac{\alpha_5}{2^2\cdot 4^2},\\
&\alpha_4^*=-\left(\frac{2\alpha_1^*}{6^2\cdot 8}+\frac{2\alpha_2^*}{4^2\cdot 6}+\frac{2\alpha_3^*}{2^2\cdot 4}\right),\\
&\alpha_5^*=\frac{3\alpha_1^*}{6^2\cdot 8^2}+\frac{2\alpha_2^*}{4^2\cdot 6^2}+\frac{\alpha_3^*}{2^2\cdot 4^2}.
\end{aligned}\right\}\tag{39}$$

由 q 的四次項及 w_1, w_3, u_2, 得

$$ru_4''+u_4'-\frac{u_4}{r}=(\beta_1r^{14}+\beta_2r^{12}+\cdots+\beta_7r^2)+\frac{4h^4}{3R^4}(\beta_1^*r^{10}+\beta_2^*r^8+\cdots+\beta_5^*r^2),\tag{40}$$

其中

$$
\left.\begin{aligned}
\beta_1 &= -\frac{\alpha_1}{4\cdot 10^2\cdot 32}(15-\nu),\\
\beta_2 &= -\frac{3\alpha_2}{20\cdot 32^2}(13-\nu)+\frac{\alpha_1}{4\cdot 10^2\cdot 32}(13-\nu),\\
\beta_3 &= -\frac{\alpha_3}{3\cdot 32^2}(11-\nu)+\frac{3\alpha_2}{20\cdot 32^2}(11-\nu),\\
\beta_4 &= -\frac{3\alpha_4}{8\cdot 32^2}(9-\nu)+\frac{\alpha_3}{3\cdot 32^2}(9-\nu),\\
\beta_5 &= -\frac{3\alpha_5}{16\cdot 32}(7-\nu)+\frac{\alpha_4}{32^2}(7-\nu),\\
\beta_6 &= -\frac{3\alpha_6}{64}(5-\nu)+\frac{3\alpha_5}{16\cdot 32}(5-\nu),\\
\beta_7 &= \frac{3\alpha_6}{64}(3-\nu),\\
\beta_1^* &= \frac{18}{32^5}a_1(1358\nu-699)-\frac{9b_1}{8\cdot 32^4}(121\nu-331),\\
\beta_2^* &= -\frac{9}{4\cdot 32^4}a_1(1031\nu-565)+\frac{9b_1}{4\cdot 32^4}(221\nu-535),\\
\beta_3^* &= \frac{9a_1}{4\cdot 32^4}(1125\nu-675)-\frac{9b_1}{4\cdot 32^4}(483\nu-633),\\
\beta_4^* &= -\frac{9a_1}{8\cdot 32^4}(969\nu-802)+\frac{9b_1}{8\cdot 32^4}(357\nu-665),\\
\beta_5^* &= \frac{27a_1}{16\cdot 32^4}(96\nu-109)-\frac{9b_1}{8\cdot 32^4}(72\nu-137).
\end{aligned}\right\} \tag{41}
$$

邊界條件爲

$$
u_4\Big|_{r=0} = 有限, \qquad u_4\Big|_{r=1} = 0, \tag{42}
$$

解得

$$
\begin{aligned}
u_4 = {} & \left(\frac{\beta_1}{14\cdot 16}r^{15}+\frac{\beta_2}{12\cdot 14}r^{13}+\cdots+\frac{\beta_7}{2\cdot 4}r^3+\frac{\beta_8}{2}r\right)+\\
& +\frac{4h^4}{3R^4}\left(\frac{\beta_1^*}{10\cdot 12}r^{11}+\frac{\beta_2^*}{8\cdot 10}r^9+\cdots+\frac{\beta_5^*}{2\cdot 4}r^3+\frac{\beta_6^*}{2}r\right),
\end{aligned} \tag{43}
$$

其中

$$
\beta_8 = -2\left(\frac{\beta_1}{14\cdot 16}+\cdots+\frac{\beta_7}{2\cdot 4}\right), \quad \beta_6^* = -2\left(\frac{\beta_1^*}{10\cdot 12}+\cdots+\frac{\beta_5^*}{2\cdot 4}\right). \tag{44}
$$

以後的計算可以仿此，我們只取以上的結果作爲近似解.

將(28)代入(13)，注意其中 α, ϕ 由(26)確定，可得應力的表達式:

$$
\sigma_r = \sigma_{r1}z+\sigma_{r2}+\sigma_{r3}z^3+\sigma_{r4}z^2, \quad \sigma_t = \sigma_{t1}z+\sigma_{t2}+\sigma_{t3}z^3+\sigma_{t4}z^2; \tag{45}
$$

其中

$$
\left.\begin{aligned}
\sigma_{r1} &= -\left(w_1'' + \nu \frac{w_1'}{r}\right) q - \left(w_3'' + \nu \frac{w_3'}{r}\right) q^3 + \cdots, \\
\sigma_{r2} &= \left(u_2' - \nu \frac{u_2}{r} + \frac{1}{2} w_1'^2\right) q^2 + \left(u_4' + \nu \frac{u_4}{r} + w_1' w_3'\right) q^4 + \cdots, \\
\sigma_{r3} &= \frac{4h^4}{3R^4}\left(-b_1 \frac{w_1'}{r} + a_1 w_1''\right) E_{12} q^3 + \cdots, \\
\sigma_{r4} &= \frac{4h^4}{3R^4}\left[\left(b_1 \frac{u_2}{r} - a_1 u_2' - \frac{1}{2} a_1 w_1'^2\right) E_{12} + \left(-b_1 \frac{w_1'}{r} + a_1 w_1''\right) E_{23}\right] q^4 + \cdots; \\
\sigma_{t1} &= -\left(\frac{w_1'}{r} + \nu w_1''\right) q - \left(\frac{w_3'}{r} + \nu w_3''\right) q^3 + \cdots, \\
\sigma_{t2} &= \left(\frac{u_2}{r} + \nu u_2' + \frac{1}{2} \nu w_1'^2\right) q^2 + \left(\frac{u_4}{r} + \nu u_4' + \nu w_1' w_3'\right) q^4 + \cdots, \\
\sigma_{t3} &= \frac{4h^4}{3R^4}\left(-b_1 w_1'' + a_1 \frac{w_1'}{r}\right) E_{12} q^3 + \cdots, \\
\sigma_{t4} &= \frac{4h^4}{3R^4}\left[\left(b_1 u_2' + \frac{1}{2} b_1 w_1'^2 - a_1 \frac{u_2}{r}\right) E_{12} + \left(-b_1 w_1'' + a_1 \frac{w_1'}{r}\right) E_{23}\right] q^4 + \cdots.
\end{aligned}\right\} \quad (46)
$$

其中 E_{12}, E_{23} 是將(28)代入(22)求得的 q 的二次及三次項：

$$
\left.\begin{aligned}
E_{12} &= \frac{9}{32^2}(13r^4 - 12r^2 + 3), \\
E_{23} &= -\frac{18}{32^4}[3(9\nu - 7)r^8 - 4(23\nu - 19)r^6 + 6(19\nu - 17)r^4 - \\
&\quad - (54\nu - 71)r^2 + 3(3\nu - 5)].
\end{aligned}\right\} \quad (47)
$$

以上得到的結果是：(28)給出板的位移，(45)給出板的應力，其中 w_i, u_i 由(31)、(34)、(38)、(43)給定.

應當指出 $w_1 q$ 即爲彈性圓板小撓度的解，(38)式 w_3 分兩部分：用 w_3^e 表前一括號部分，w_3^p 表後一括號部分，則 $w_1 q + w_3^e q^3$ 即爲彈性圓板大撓度的解，$w_1 q + w_3^p q^3$ 即爲塑性圓板小撓度的解[2]. 由此可知，彈塑性圓板大撓度的近似解可以彈性小撓度的解爲基礎，分別疊加大撓度及材料塑性的影響. 材料塑性的影響是 $w_3^p q^3$，它和 $\frac{h^4}{R^4}$ 成正比. 以上所指是撓度，至於應力或其他則不然了.

小彈塑性形變理論要求主應力方向不變、應力分量比不變. 這裏僅檢查兩點：在板心顯然如此；在板邊例如 $r = 1$, $z = \pm 1$ 處主應力方向不變，應力分量比由(6)，因 $e_t = 0$ 得 $\frac{\sigma_t}{\sigma_r} = \alpha$，而 α 値由 ν 到 $\frac{1}{2}$，這個變化可以認爲不大.

四. 一般應力-應變關係的解法

對於任意的廣義應力-應變關係，用前節小參數法常有困難，本節採用漸近法求解，不用無量綱量. 將(10)展成 $\omega(e_i)$ 的冪級數，可得和(18)相似的結果：

$$\left.\begin{aligned} \alpha &= \nu + \nu_1\omega(e_i) + \cdots, & \nu_1 &= \frac{1}{3}(1+\nu)(1-2\nu), \\ \phi &= \frac{E}{1-\nu^2}[1 - a_1\omega(e_i) + \cdots], & a_1 &= \frac{2}{3}\frac{1-\nu+\nu^2}{1-\nu}, \\ \phi\alpha &= \frac{E}{1-\nu^2}[\nu + b_1\omega(e_i) + \cdots], & b_1 &= \frac{1-4\nu+\nu^2}{3(1-\nu)}. \end{aligned}\right\} \tag{48}$$

僅取級數的前兩項,代入(15),得

$$\left.\begin{aligned} J_1 &= \frac{E}{1-\nu^2}[2h - a_1\Omega_1], \ J_2 = \frac{E}{1-\nu^2}[-a_1\Omega_2], \ J_3 = \frac{E}{1-\nu^2}\left[\frac{2}{3}h^3 - a_1\Omega_3\right]; \\ I_1 &= \frac{E}{1-\nu^2}[2h\nu + b_1\Omega_1], I_2 = \frac{E}{1-\nu^2}[b_1\Omega_2], \quad I_3 = \frac{E}{1-\nu^2}\left[\frac{2}{3}h^3\nu + b_1\Omega_3\right]; \\ \Omega_1 &= \int_{-h}^{+h}\omega(e_i)dz, \quad \Omega_2 = \int_{-h}^{+h}\omega(e_i)zdz, \quad \Omega_3 = \int_{-h}^{+h}\omega(e_i)z^2dz. \end{aligned}\right\} \tag{49}$$

給定廣義應力-應變關係,則 Ω_i 可求得爲 w, u 的函數,例如廣義應力-應變關係用冪級數表示時 Ω_i 可由(24)式得到. 對綫性硬化的情形 Ω_i 也有算式表示成 w, u 的函數.

將(49)代入(16),略加簡化得

$$ru'' + u' - \frac{u}{r} = F_1, \tag{50a}$$

$$rw''' + w'' - \frac{w'}{r} = F_2. \tag{50b}$$

其中

$$\left.\begin{aligned} F_1 = \frac{1}{2h}\Big\{&-2h\left(\frac{1}{2}w'^2 + rw'w'' - \frac{1}{2}\nu w'^2\right) + \\ &+\left[\left(u''r + u' - \frac{u}{r} + rw'w'' + \frac{1}{2}w'^2\right)a_1 + \frac{1}{2}w'^2b_1\right]\Omega_1 - \\ &-\left(w'''r + w'' - \frac{w'}{r}\right)a_1\Omega_2 + \left(ru' + \frac{1}{2}rw'^2\right)a_1\Omega_1' - \\ &-ub_1\Omega_1' - (w''ra_1 - w'b_1)\Omega_2'\Big\}, \\ F_2 = \frac{3}{2h^3}\Big\{&\left(rw'u' + \frac{1}{2}w'^3r\right)(2h - a_1\Omega_1) + w'u(2h\nu + b_1\Omega_1) - \\ &-\left[\left(u''r + u' - \frac{u}{r} + \frac{1}{2}w'^2\right)a_1 + \frac{3}{2}w'^2b_1\right]\Omega_2 + \\ &+\left(w'''r + w'' - \frac{w'}{r}\right)a_1\Omega_3 - \left[\left(ru' + \frac{1}{2}rw'^2\right)a_1 - b_1u\right]\Omega_2' + \\ &+(rw''a_1 - w'b_1)\Omega_3' + \frac{(1-\nu^2)}{2E}qr^2\Big\}. \end{aligned}\right\} \tag{51}$$

(50)可寫成

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = F_1, \ r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rw')\right] = F_2. \tag{52}$$

積分並用周邊固定的邊界條件

$$u\Big|_{r=0}=0,\quad u\Big|_{r=R}=0,\quad w'\Big|_{r=0}=0,\quad w'\Big|_{r=R}=0, \tag{53}$$

得

$$\left.\begin{aligned} u&=\frac{r}{2}\int_0^r\frac{F_1}{r}dr-\frac{1}{2r}\int_0^r rF_1dr+\frac{r}{2R^2}\int_0^R rF_1dr-\frac{r}{2}\int_0^R\frac{F_1}{r}dr,\\ w'&=\frac{r}{2}\int_0^r\frac{F_2}{r}dr-\frac{1}{2r}\int_0^r rF_2dr+\frac{r}{2R^2}\int_0^R rF_2dr-\frac{r}{2}\int_0^R\frac{F_2}{r}dr.\end{aligned}\right\} \tag{54}$$

利用(54)即可漸近地求 u, w'. 任意取一組近似的 u, w', 由(22), (21)求 E_1, E_2, E_3, e_i; 由(49)求 $\Omega_1, \Omega_2, \Omega_3$; 由(51)求 F_1, F_2, 於是由(54)得進一步的 u 和 w'.

第一步的近似值可用彈性小撓度或大撓度的解. 對於廣義應力-應變關係可用冪級數表示的情形,每一步的近似解都有分析式表示,至於其他應力-應變關係,若(54)右邊不能直接積分,則可用數值積分代替.

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ON THE LARGE DEFLECTION OF ELASTO-PLASTIC BENDING OF CIRCULAR PLATES

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ABSTRACT

In this paper the problem of large deflection of elasto-plastic bending of an uniformly loaded circular plate is studied and the solution is obtained in explicit expressions by the method of perturbation.

The stress-strain relations may be derived from the theory of small elasto-plastic deformation and in the case of circular plates the radial and tangential stress are as follows:

$$\left.\begin{aligned}\sigma_r &= \phi(e_r + \alpha e_t),\\ \sigma_t &= \phi(e_t + \alpha e_r),\end{aligned}\right\} \tag{6}$$

where e_r and e_t are the radial and tangential strains respectively; ϕ, the elasto-plastic coefficient defined by

$$\phi = \frac{2}{3}\frac{\sigma_i}{e_i}\frac{2K + \frac{2}{9}\frac{\sigma_i}{e_i}}{K + \frac{4}{9}\frac{\sigma_i}{e_i}}, \tag{7}$$

and α, the coefficient of lateral deformation defined by

$$\alpha = \frac{K - \frac{2}{9}\frac{\sigma_i}{e_i}}{2K + \frac{2}{9}\frac{\sigma_i}{e_i}}. \tag{7}$$

Here K is the bulk modulus; σ_i, the intensity of stress given by

$$\sigma_i = \sqrt{\sigma_r^2 - \sigma_r\sigma_t + \sigma_t^2}, \tag{5}$$

and e_i, the intensity of strain given by

$$e_i = \frac{2}{\sqrt{3}}\sqrt{e_r^2 + e_r e_t + e_t^2}.$$

Using the above relations and the relations between the strains and the components of displacement, the equations of equilibrium for this problem reduce to

$$\left.\begin{aligned}&-\left(w'''r + w'' - \frac{w'}{r}\right)J_2 + \left(u''r + u' - \frac{u}{r} + \frac{1}{2}w'^2 + rw'w'''\right)J_1 -\\ &-\frac{1}{2}w'^2 I_1 - w''rJ_2' - w'I_2' + \left(u'r + \frac{1}{2}w'^2 r\right)J_1' + uI_1' = 0,\\ &-\left(w'''r + w'' - \frac{w'}{r}\right)J_3 + \left(u''r + u' - \frac{u}{r} + \frac{1}{2}w'^2\right)J_2 - \frac{3}{2}w'^2 I_2 +\\ &+\left(w'u'r + \frac{1}{2}w'^3 r\right)J_1 + w'uI_1 - w''rJ_3' + \left(u'r + \frac{1}{2}w'^2 r\right)J_2' -\\ &- w'I_3' + uI_2' + \frac{qr^2}{2} = 0,\end{aligned}\right\} \tag{16}$$

where w and u are components of displacement; $w' = \frac{dw}{dr}$, $u'' = \frac{d^2u}{dr^2}$, …etc.; r, the radius vector, and the $J's$ and $I's$ are defined as follows:

$$\left.\begin{array}{lll} J_1 = \int_{-h}^{+h} \phi dz; & J_2 = \int_{-h}^{+h} \phi z dz; & J_3 = \int_{-h}^{+h} \phi z^2 dz; \\ I_1 = \int_{-h}^{+h} \phi \alpha dz; & I_2 = \int_{-h}^{+h} \phi \alpha z dz; & I_3 = \int_{-h}^{+h} \phi \alpha z^2 dz; \end{array}\right\} \tag{15}$$

where h is the half of the thickness of the plate; z, the vertical coordinate.

Suppose that the σ_i–e_i relation can be expressed by a power series as proposed by D. Trifan[2]:

$$\sigma_i = 3G(1 - \gamma_1 e_i^2)e_i,$$

where G and γ_1 are the shear modulus and a constant respectively; and suppose that

$$w = w_1 q + w_3 q^3 + \cdots, \quad u = u_2 q^2 + u_4 q^4 + \cdots,$$

where q is the intensity of uniform load and $w_1, w_3, \cdots, u_2, u_4, \cdots$ are functions of r only. Substituting them in the basic equations (16) and equating the coefficients of the same power of q, we obtain a group of linear equations for w_1, $w_3, \cdots, u_2, u_4, \cdots$, which can be easily solved. The final results are then the combinations of $w_1, w_3, \cdots, u_2, u_4, \cdots$.

When the σ_i–e_i relation takes a general form, an approximate method is proposed for this problem.