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研究簡报

关于一个表面波問題里的積分

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刘先志先生在德國1952年的力学学报上發表的"Über die Entstehung von Ringwellen an einer Flüssigkeits oberfläche durch unter dieser gelegene, Kugelige periodische Quellensysteme, ZAMM Juli 1952"文中,有一積分如下:

$$I=\int_0^\infty e^{-(h+z)\xi}\frac{\xi J_0(\xi\gamma)}{\xi-\dfrac{\omega^2}{g}}d\xi. \tag{1}$$

为了研究在泉湧远处的波形,我們要求出(1)式積分当$\gamma\gg1$时的值. 为了便于積分,刘先生用零階貝塞尔函数$J_0(\xi\gamma)$的漸近值,把上式变成

$$I=\sqrt{\frac{2}{\pi\gamma}}\int_0^\infty e^{-(h+z)\xi}\frac{\sqrt{\xi}\cos\left(\xi\gamma-\dfrac{\pi}{4}\right)}{(\xi-\omega^2/g)}d\xi.$$

但是当变数ξ很小时,虽然$\gamma\gg1$,还是不能用漸近值來代替$J_0(\xi\gamma)$. 因此用上面的方法計算引進了誤差. 本文作者建議用下面方法計算(1)式的積分:令

$$I_1=\int_0^\infty e^{-(h+z)\xi}\frac{J_0(\xi\gamma)}{\xi-\dfrac{\omega^2}{g}}d\xi,$$

代入(1)式,則

$$I=\int_0^\infty e^{-(h+z)\xi}J_0(\xi\gamma)d\xi+\frac{\omega^2}{g}I_1=\frac{1}{\sqrt{\gamma^2+(h+z)^2}}+\frac{\omega^2}{g}I_1. \tag{2}$$

引進$J_0(\xi\gamma)=\dfrac{2}{\pi}\displaystyle\int_1^\infty\frac{\sin\xi\gamma t}{\sqrt{t^2-1}}dt$,代入上面的積分$I_1$,并先对$\xi$積分,則得

$$I_1=\frac{1}{\pi}\int_1^\infty \frac{dt}{\sqrt{t^2-1}}(I_2-I_3). \tag{3}$$

此处

$$I_2=\frac{1}{i}\int_0^\infty e^{[-(h+z)+i\gamma t]\xi}\frac{d\xi}{\xi-\frac{\omega^2}{g}}; \tag{4}$$

$$I_3=\frac{1}{i}\int_0^\infty e^{-[(h+z)+i\gamma t]\xi}\frac{d\xi}{\xi-\frac{\omega^2}{g}}. \tag{5}$$

若采取如圖(1)所示的路綫求 I_2，当 $R\to\infty$，則

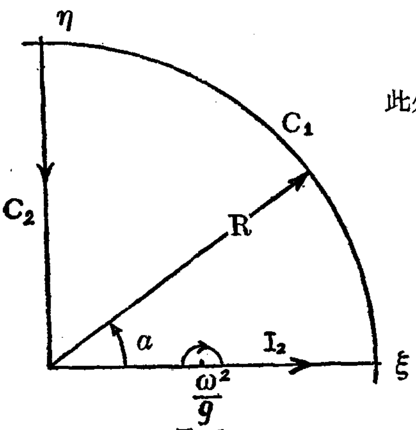

圖 1.

$$I_2+C_1+C_2-\pi e^{[-(h+z)+i\gamma t]\omega^2/g}=0, \tag{6}$$

此处

$$C_1=\lim_{R\to\infty}\int_0^{\frac{\pi}{2}} e^{[-(h+z)+i\gamma t]Re^{i\theta}}\frac{Re^{i\theta}d\theta}{Re^{i\theta}-\frac{\omega^2}{g}}; \tag{7}$$

$$C_2=\frac{1}{i}\int_{i\infty}^0 e^{[-(h+z)+i\gamma t]\xi}\frac{d\xi}{\xi-\frac{\omega^2}{g}}. \tag{8}$$

从(7)式，得

$$C_1=\lim_{R\to\infty}\int_0^{\frac{\pi}{2}} e^{-[(h+z)\cos\theta+\gamma t\sin\theta]R}\, e^{i[-(h+z)\sin\theta+\gamma t\cos\theta]R}\frac{Re^{i\theta}}{Re^{i\theta}-\frac{\omega^2}{g}}d\theta.$$

設 M 为 $(h+z)\cos\theta+\gamma t\sin\theta$ 在 $0\leqslant\theta\leqslant\frac{\pi}{2}$ 的最小值．因 $h+z>0$，$\gamma>0$，$t>0$；$0\leqslant\theta\leqslant\frac{\pi}{2}$，故 M 为一非零的正値数．于是

$$|C_1|\leqslant\lim_{R\to\infty}\frac{\pi}{2}e^{-MR}=0; \tag{9}$$

因而

$$C_1=0.$$

引進 $\xi=is$，代入(8)式，得

$$C_2=-\int_0^\infty e^{-[i(h+z)+\gamma t]s}\frac{ds}{is-\frac{\omega^2}{g}}. \tag{10}$$

將 (9) 及 (10) 式的結果代入 (6) 式,得

$$I_2=\int_0^\infty e^{-[i(h+z)+\gamma t]s}\frac{ds}{is-\frac{\omega^2}{g}}+\pi e^{[-(h+z)+i\gamma t]\omega^2/g}. \qquad (11)$$

同样的,可以按圖 (2) 所示的路綫求得:

$$I_3=\int_0^\infty e^{[i(h+z)-\gamma t]s}\frac{ds}{is+\frac{\omega^2}{g}}-\pi e^{-[(h+z)+i\gamma t]\omega^2/g}. \qquad (12)$$

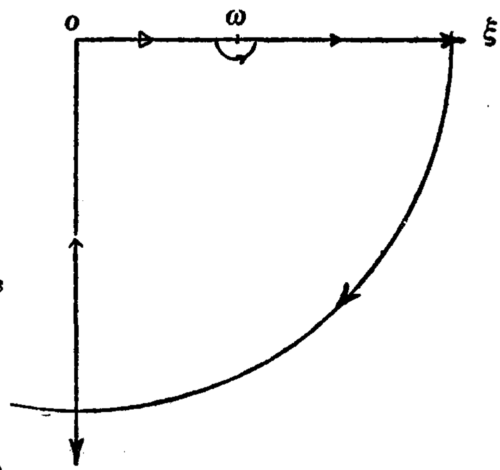

圖 2.

从 (11) 及 (12) 式,得

$$I_2-I_3=-2\int_0^\infty e^{-\gamma st}\frac{1}{s^2+\frac{\omega^4}{g^2}}\left[\frac{\omega^2}{g}\cos(h+z)s+s\sin(h+z)s\right]ds+2\pi e^{-(h+z)\omega^2/g}\cos\frac{\omega^2}{g}\gamma t. \qquad (13)$$

將 (13) 式代入 (3) 式,对 t 積分,并用下列关系:

$$K_0(z)=\int_1^\infty\frac{e^{-zt}}{\sqrt{t^2-1}}\alpha t;\qquad Y_0(z)=-\frac{2}{\pi}\int_1^\infty\frac{\cos zt}{\sqrt{t^2-1}}dt,$$

得

$$I_1=-\frac{2}{\pi}\int_0^\infty\frac{K_0(\gamma s)}{s^2+\frac{\omega^4}{g^2}}\left[\frac{\omega^2}{g}\cos(z+h)s+\sin(h+z)s\right]ds-\pi e^{-(h+z)\omega^2/g}Y_0\left(\frac{\omega^2}{g}\gamma\right). \qquad (14)$$

令 $\gamma s=\eta$, $a=\frac{\omega^2(h+z)}{g}$, $\epsilon=\frac{g}{\gamma\omega^2}$, 代入上式中的積分,并令

$$I_4(\epsilon)=\int_0^\infty\frac{K_0(\gamma s)}{s^2+\frac{\omega^4}{g^2}}\left[\frac{\omega^2}{g}\cos(z+h)s+s\sin(h+z)s\right]ds,$$

故

$$I_4(\epsilon)=\epsilon\int_0^\infty\frac{K_0(\eta)}{\epsilon^2\eta^2+1}[\cos a\epsilon\eta+\epsilon\eta\sin a\epsilon\eta]d\eta. \qquad (15)$$

则我們可以把 $I_4(\epsilon)$ 展开成 ϵ 的級数如下:

$$I_4(\epsilon)=I_4(0)+\epsilon I_4^{(1)}(0)+\frac{\epsilon^2 I_4^{(2)}(0)}{2!}+\cdots\cdots+\frac{\epsilon^n I_4^{(n)}(0)}{n!}+\frac{\epsilon^{n+1}I_4^{(n+1)}(0)}{(n+1)!}. \qquad (16)$$

此处 $0\leqslant\overline{\theta}\leqslant\epsilon$ 及 $I_4^{(n)}=\frac{d^n I_4}{d\epsilon^n}$. 故

$$I_4(\epsilon)=\epsilon\int_0^\infty K_0(\eta)\left[1-\left(1-a+\frac{a^2}{2!}\right)\epsilon^2\eta^2+\left(1-a+\frac{a^2}{2!}-\frac{a^3}{3!}+\frac{a^4}{4!}\right)\epsilon^4\eta^4\right.$$

$$\left.+\cdots\cdots(-1)^M\left(1+\sum_{K=1}^{2M}(-1)^K\frac{a^K}{K!}\right)(\epsilon\eta)^{2M}\right]\lambda\eta+\frac{I_4^{(2M+2)}(\overline{\theta})\epsilon^{2M+2}}{(2M+2)!}$$

$$=\epsilon\left[\frac{1}{2}\Gamma^2\left(\frac{1}{2}\right)-\left(1-a+\frac{a^2}{2!}\right)2\epsilon^2\Gamma^2(3/2)+\left(1-a+\frac{a^2}{2!}-\frac{a^3}{3!}\right.\right.$$

$$\left.\left.+\frac{a^4}{4!}\right)2^3\Gamma^2(5/2)\epsilon^4+\cdots\cdots+(-1)^M\left(1+\sum_{K=1}^{2M}(-1)^K\frac{a^K}{K!}\right)\epsilon^{2M}\Gamma^2\left(\frac{2M+1}{2}\right)2^{2M-1}\right]$$

$$+\frac{I_4^{(2M+2)}(\overline{\theta})\epsilon^{2(M+1)}}{(2M+2)!}. \qquad (17)$$

(17) 式是一漸近級数，依据 ϵ 的大小，用絕对値最小的一項以前的級数來代表 $I_4(\epsilon)$. 設 $I_4^{(2M+2)}(\overline{\theta})$ 的最大値是 N，如果用 (17) 式中括弧內的級数來代表 $I_4(\epsilon)$，其誤差是小于或者等于 $\frac{N\epsilon^{2(M+1)}}{(2M+2)!}$.

將 (17) 式代入 (14) 式，求出 I_1，再將 I_1 代入 (2) 式，得到 $I(\epsilon)$ 的漸近級数如下：

$$I=\frac{\omega^2}{g}\left[\frac{\epsilon}{\sqrt{1+a^2\epsilon^2}}-\pi e^{-a}Y_0\left(\frac{1}{\epsilon}\right)+\left\{-\epsilon+\left(1-a+\frac{a^2}{2!}\right)\epsilon^3\right.\right.$$

$$-\left(1-a+\frac{a^2}{2!}-\frac{a^3}{3!}+\frac{a^4}{4!}\right)\frac{(3!)^2}{2^2}\epsilon^5+\cdots\cdots$$

$$\left.\left.+(-1)^{M+1}\left(1+\sum_{K=1}^{2M}(-1)^K\frac{a^K}{K!}\right)\frac{[(2M-1)!]^2\epsilon^{2M+1}}{2^{2(M-1)}[(M-1)!]^2}\right\}\right].$$

本文經力学研究所陈小仲同志校对核算，并改正一些錯誤，謹致謝意.

ABOUT AN INTEGRAL OF SURFACE WAVES

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ABSTRACT

In his paper[1)], Prof. H. C. Liu calculated an integral

$$I=\int_0^\infty e^{(h+z)\xi}\frac{\xi J_0(\xi\gamma)}{\xi-\frac{\omega^2}{g}}d\xi. \tag{1}$$

Since γ is very large, he replaced $J_0(\xi\gamma)$ by its asymptotic form, the above integral, therefore, is reduced to:

$$I=\int_0^\infty e^{-(h+z)\xi}\frac{\sqrt{\xi}\cos\left(\xi\gamma-\frac{\pi}{4}\right)}{\xi-\frac{\omega^2}{g}}d\xi. \tag{2}$$

However, inspite of that $\gamma\gg 1$, $J_0(\xi\gamma)$ can not be replaced asymptotically, when ξ is small. In the present note, it is suggested that, without using the asymptotic form of $J_0(\xi\gamma)$, an asymptotic series of the integral (1) is obtained as follows:

An introduction into expression (1) of $J_0(\xi\gamma)=\frac{1}{\pi}\int_1^\infty\frac{e^{i\xi\gamma t}-e^{-i\xi\gamma t}}{i\sqrt{t^2-1}}dt$ yields:

$$I=\int_0^\infty e^{-(h+z)\xi}J_0(\xi\gamma)d\xi+\frac{\omega^2}{\pi g}\int_1^\infty\frac{dt}{\sqrt{t^2-1}}(I_1-I_2),$$

where

$$\left.\begin{aligned} I_1&=\frac{1}{i}\int_0^\infty e^{(-h-z+i\gamma t)\xi}\frac{d\xi}{\xi-\frac{\omega^2}{g}},\\ I_2&=\frac{1}{i}\int_0^\infty e^{(-h-z-i\gamma t)\xi}\frac{d\xi}{\xi-\frac{\omega^2}{g}}.\end{aligned}\right\} \tag{3}$$

1) H. C. Liu, "Über die Entstehung von Ringwellen an einer Flüssigkeits oberfläche durch unter dieser gelegene, Kugelige periodische Quellensysteme", ZAMM Juli 1952.

Following the paths shown in Figs. 1 & 2, I_1 and I_2 can be calculated respectively by contour integration and it results in:

$$I=\int_0^\infty e^{-(h+z)\xi}J_0(\xi\gamma)d\xi-\frac{2\omega^2}{\pi g}\int_0^\infty \frac{K_0(\gamma t)}{t^2+\frac{\omega^4}{g^2}}\left[\frac{\omega^2}{g}\cos(h+z)t+t\sin(h+z)t\right]dt$$

$$-\frac{\omega^2}{g}\pi e^{-(h+z)\omega^2/g}\,Y_0\left(\frac{\omega^2\gamma}{g}\right). \qquad (4)$$

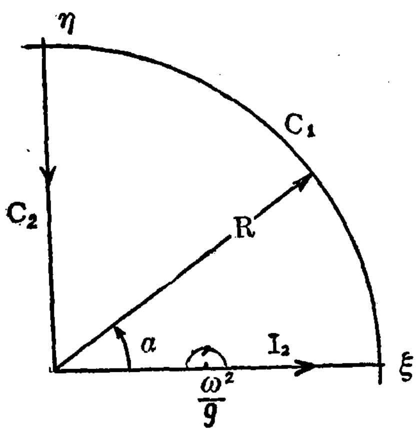

Fig. 1

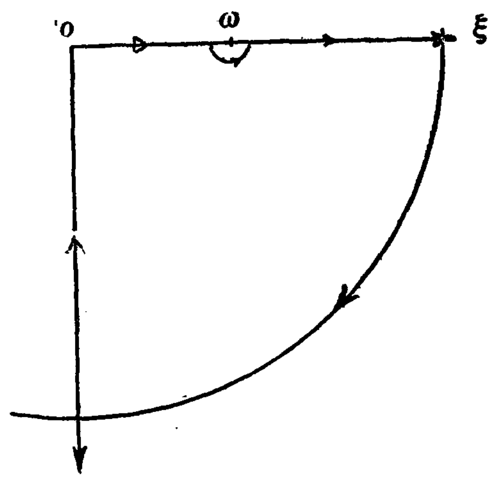

Fig. 2

Inserting $\gamma t=\eta$, $a=\frac{\omega^2(h+z)}{g}$ and $\epsilon=\frac{g}{\gamma\omega^2}$ into expression (4), we have

$$I=\frac{\omega^2}{g}\left[\frac{\epsilon}{\sqrt{1+a^2\epsilon^2}}-\pi e^{-a}Y_0\left(\frac{1}{\epsilon}\right)-\frac{2}{\pi}\epsilon\int_0^\infty\frac{K_0(\eta)}{\epsilon^2\eta^2+1}(\cos a\epsilon\eta+\epsilon\eta\sin a\epsilon\eta)d\eta\right]. \qquad (5)$$

The integral in eq.(5) may be taken as a function of ϵ, and expanded into an asymptotic series as such

$$I_3=\int_0^\infty\frac{K_0(\eta)}{\epsilon^2\eta^2+1}(\cos a\epsilon\eta+\epsilon\eta\sin a\epsilon\eta)d\eta$$

$$=I_3(0)+\epsilon I_3^{(1)}(0)+\frac{\epsilon^2}{2!}I_3^{(2)}(0)+\cdots\cdots+\frac{\epsilon^n I_3^{(n)}(0)}{n!}+\epsilon^{n+1}\frac{I_3^{(n+1)}(\bar\theta)}{(h+1)!},$$

where

$$0\leqslant\bar\theta\leqslant\epsilon,\quad \text{and}\quad I_3^{(n)}=\frac{d^nI_3}{d\epsilon^n}. \qquad (6)$$

It is easy to calculate $I_3^{(n)}(0)$ from eq.(6). Consequently the asymptotic series of I may be written as:

$$I=\frac{\omega^2}{g}\left[\frac{\epsilon}{\sqrt{1+a^2\epsilon^2}}-\pi e^{-a}Y_0\left(\frac{1}{\epsilon}\right)+\left\{-\epsilon+\left(1-a+\frac{a^2}{2!}\right)\epsilon^3-\left(1-a+\frac{a^2}{2!}-\frac{a^3}{3!}+\frac{a^4}{4!}\right)\frac{(3!)^2}{2^2}\epsilon^5+\cdots\cdots+(-1)^{M+1}\left(1+\sum_{K=1}^{2M}(-1)^K\frac{a^K}{K!}\right)\frac{[(2M-1)!]^2\epsilon^{2M+1}}{2^{2(M-1)}[(M-1)!]^2}\right\}\right].$$

Where the error is equal to or less than $\left|\frac{\epsilon^{M+2}I^{(2M+1)}(\bar{\theta})}{(2M+1)!}\ \frac{2}{\pi}\right|$, and the series ends before the term, with the smallest absolute value.